Journal of Statistical Physics, Vol. 98, Nos. 5/6, 2000

A Modified Boltzmann Equation for Bose–Einstein Particles: Isotropic Solutions and Long-Time Behavior

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Received August 4, 1999; final November 10, 1999

Under some strong cutoff conditions on collision kernels, global existence, local stability, entropy identity, conservation of energy, and moment production estimates are proven for isotropic solutions of a modified (quantum effect) Boltzmann equation for spatially homogeneous gases of Bose–Einstein particles (BBE). Then applying these results with the biting-weak convergence, some results on the long-time behavior of the conservative isotropic solutions of the BBE equation are obtained, including the velocity concentration at very low temperatures and the tendency toward equilibrium states at very high temperatures.

KEY WORDS: Modified Boltzmann equation; Bose–Einstein particles; quantum effect; entropy identity; temperature condition; velocity concentration; equilibrium; biting-weak convergence.

1. INTRODUCTION

We study time-evolution of spatially homogeneous gases of Bose–Einstein identical particles governed by the following modified Boltzmann equation which takes a quantum effect into account:

$$\frac{\partial}{\partial t} f(v,t) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times [f' f'_* (1 + \varepsilon f)(1 + \varepsilon f_*) - f f_* (1 + \varepsilon f')(1 + \varepsilon f'_*)]\, d\omega\, dv_* \tag{BBE}$$

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0022-4715/00/0300-1335$18.00/0

where $B(z, \omega)$ is the collision kernel which is a nonnegative Borel function of $|z|$, $|\langle z, \omega\rangle|$ only:

$$B(z, \omega) \equiv \bar{B}\left(|z|, \frac{|\langle z, \omega\rangle|}{|z|}\right), \qquad (z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2$$

The solutions f are velocity distribution functions (i.e., the density functions of particle number), $\varepsilon = (h/m)^3/\mathrm{g}$, h is the Planck's constant, m and g are the mass and the "statistical weight" of a particle.

From Chapman and Cowling (ref. 7, Chap. 17) we know that Eq. (BBE) is established on the basis of the following argument: When the mean distance between neighbouring molecules is comparable with the size of the quantum wave fields with which molecules are surrounded, a state of congestion results. For a gas composed of Bose–Einstein identical particles, according to quantum theory, the presence of a like particle in the velocity-range dv increases the probability that a particle will enter that range; the presence of $f(v)\, dv$ particles per unit volume increases this probability in the ratio $1 + \varepsilon f(v)$. This yields the Eq. (BBE).

Symmetrically, replacing the "increases" and the ratio "$1 + \varepsilon f$" with "decreases" and "$1 - \varepsilon f(v)$" leads to the following modified Boltzmann equation for Fermi–Dirac identical particles (due to the Pauli exclusion principle):

$$\frac{\partial}{\partial t} f(v, t) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times [f' f'_* (1 - \varepsilon f)(1 - \varepsilon f_*) - f f_* (1 - \varepsilon f')(1 - \varepsilon f'_*)]\, d\omega\, dv_* \tag{BFD}$$

Of course, if the quantum effect is not taken into account, i.e., if $\varepsilon = 0$, we go back to the original Boltzmann equation

$$\frac{\partial}{\partial t} f(v, t) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)(f' f'_* - f f_*)\, d\omega\, dv_* \tag{B}$$

In statistical physics and experiments, it is well-known that the evolution of the Bose–Einstein particles at very low temperatures exhibits the Bose–Einstein condensation (i.e., the velocity concentration) which is essentially different from those of the Fermi-Dirac particles. From a mathematical viewpoint, this phenomenon of velocity concentration can be also explained as the lack of L^1-weak compactness of the function set $\{f(\cdot, t)\}_{t \geqslant 0}$, where f is a solution of Eq. (BBE). The main purpose of the present paper is to show this.

Let us first recall that for Eq. (B) and Eq. (BFD) there are no problem of such weak compactness: In the case of Eq. (B), under the initial condition $f|_{t=0}=f_0$, $\int_{\mathbf{R}^3} f_0(v)(1+|v|^2+|\log f_0(v)|)\,dv<\infty$, the Boltzmann's H-theorem and the conservation of mass and energy imply that $\int_{\mathbf{R}^3} f(v,t)(1+|v|^2+|\log f(v,t)|)\,dv$ is bounded in time on $[0,\infty)$. This gives the weak compactness of $\{f(\cdot,t)\}_{t\geqslant 0}$ in $L^1(\mathbf{R}^3)$, and the solutions $f(\cdot,t)$, as $t\to\infty$, always converge (at least weakly) in $L^1(\mathbf{R}^3)$ to the equilibrium solutions which have the unique form $f_\infty(v)=a\,e^{-b\,|v-v_0|^2}$ (see e.g., refs. 1, 5, 6, 10, and 17). In the case of Eq. (BFD), because of the physical meaning of the factor $1-\varepsilon f$, one expects that the solutions satisfy $1-\varepsilon f\geqslant 0$, i.e., $0\leqslant f(v,t)\leqslant 1/\varepsilon$ on $\mathbf{R}^3\times[0,\infty)$, at least if the initial data $f|_{t=0}=f_0$ satisfy the same constraints. In other words, L^∞-bounds are available. This together with the conservation of mass and energy gives the L^1- weak compactness of $\{f(\cdot,t)\}_{t\geqslant 0}$, and the weak limits of $f(\cdot,t_n)$ $(t_n\to\infty)$ are equilibrium solutions which may have one of the following two forms (at least):

$$\frac{ae^{-b\,|v-v_0|^2}}{1+\varepsilon a\,e^{-b\,|v-v_0|^2}} \qquad \text{and} \qquad \frac{1}{\varepsilon}\,1_{\{|v-v_0|\leqslant R\}}$$

General study of (spatially inhomogeneous) solutions of Eq. (BFD) can be found in Dolbeault[8] and P. L. Lions.[12]

For Eq. (BBE), the case is completely opposite. On one hand, the factor $1+\varepsilon f$ strengthens the non-linearity in the collision integral for large value of the solution f. On the other hand, under the only condition that the mass and energy are finite, the corresponding entropy

$$S_f(t)=\int_{\mathbf{R}^3}\left[\frac{1}{\varepsilon}(1+\varepsilon f)\log(1+\varepsilon f)-f\,\log f\right]dv$$

is also finite, and satisfies (at least formally) the entropy identity:

$$\begin{aligned} S_f(t)=S_f(0)+\tfrac{1}{4}\int_0^t d\tau\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} &B(v-v_*,\omega)\\ &\times\Gamma(f'f'_*(1+\varepsilon f)(1+\varepsilon f_*),\,ff_*(1+\varepsilon f')(1+\varepsilon f'_*))\,d\omega\,dv_*\,dv,\quad t\geqslant 0 \end{aligned} \tag{1.1}$$

where

$$\Gamma(a,b)=\begin{cases}(a-b)\log(a/b), & a>0,\quad b>0;\\ +\infty, & a>0,\ b=0\quad\text{or}\quad a=0,\ b>0;\\ 0, & a=b=0\end{cases} \tag{1.2}$$

Thus, except the conservation of the mass and energy, there are no information can be directly obtained from either the structure of Eq. (BBE) or the entropy identity (1.1) in proving the weak compactness of $\{f(\cdot, t)\}_{t\geq 0}$. Next let us observe the temperature affect on the long-time behavior of solutions of Eq. (BBE). After a velocity-translation, in this paper we always assume $v_0 = 0$, i.e., $\int_{\mathbf{R}^3} f_0(v)\, v\, dv = 0$, for instance, the initial data f_0 are isotropic functions: $f_0(v) = \bar{f}_0(|v|)$. Then the equilibrium solutions of Eq. (BBE) have the unique form:

$$\Omega_{a,b}(v) = \frac{a\, e^{-b|v|^2}}{1 - \varepsilon a\, e^{-b|v|^2}}, \qquad 0 \leq a \leq 1/\varepsilon, \quad b > 0$$

for which we have the following

Proposition 1. Let M_0, M_2 be positive real numbers. Then the following moment equation system

$$\int_{\mathbf{R}^3} \Omega_{a,b}(v)\, dv = M_0, \qquad \int_{\mathbf{R}^3} \Omega_{a,b}(v)\, |v|^2\, dv \leq M_2 \tag{1.3}$$

has a solution (a, b) satisfying $0 < a \leq 1/\varepsilon$, $b > 0$, if and only if M_0, M_2 satisfy

$$\frac{M_2}{(M_0)^{5/3}} \geq \frac{3}{2\pi} \frac{\zeta(5/2)}{[\zeta(3/2)]^{5/3}}\, \varepsilon^{2/3} \tag{1.4}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $s > 1$. Moreover, under the condition (1.4), the moment equation system

$$\int_{\mathbf{R}^3} \Omega_{a,b}(v)\, dv = M_0, \qquad \int_{\mathbf{R}^3} \Omega_{a,b}(v)\, |v|^2\, dv = M_2 \tag{1.5}$$

has a unique solution (a, b) satisfying $0 < a \leq 1/\varepsilon$, $b > 0$.

The proof of Proposition 1 will be given in Section 5. Except in this proposition, in the following we always take $M_0 = \int_{\mathbf{R}^3} f_0(v)\, dv$ and $M_2 = \int_{\mathbf{R}^3} f_0(v)\, |v|^2\, dv$. Note that since in this case M_0 is the particle number per unit space volume and $\frac{1}{2}(M_2/M_0)$ is the kinetic energy per unit mass, the condition (1.4) is equivalent to the temperature condition:

$$T \geq T_b := \frac{\zeta(5/2)}{\zeta(3/2)}\, T_c \approx 0.5134 T_c$$

where

$$T = \frac{m}{2} \cdot \frac{M_2}{M_0} \cdot \frac{2}{3k_B}$$

is the temperature (see Truesdell and Muncaster,[17] pp. 43–44), k_B is the Boltzmann's constant and (because $\varepsilon = (h/m)^3/\mathrm{g}$)

$$T_c = \frac{h^2}{2\pi m k_B} \left[\frac{M_0}{\zeta(3/2)\,\mathrm{g}} \right]^{2/3}$$

is the critical temperature derived from the classical method in statistical physics for the Bose–Einstein condensation of ideal Bose gases (see Landau and Lifshitz,[11] pp. 180–181, p. 36; or Parthia[16] p. 180 for $\mathrm{g} = 1$ (spinless)). Therefore, if T satisfies the very low temperature condition: $T < T_b$, the entropy identity (1.1) implies that the set $\{f(\cdot, t)\}_{t \geqslant 0}$ can not be weakly compact in $L^1(\mathbf{R}^3)$, or equivalently, a velocity concentration happens when time tends to infinity. However for not very low temperature conditions, it is not known whether the set $\{f(\cdot, t)\}_{t \geqslant 0}$ is always weakly compact in $L^1(\mathbf{R}^3)$. But for some very high temperature conditions: $T/T_b \gg 1$, the set $\{f(\cdot, t)\}_{t \geqslant 0}$ can be weakly compact and the solution $f(\cdot, t)$ tends to the unique equilibrium solution $\Omega_{a,b}$ as $t \to \infty$. These results, including existence of solutions, will be rigorously proven in this paper for isotropic solutions of Eq. (BBE) of non-soft potential models under the following strong cutoff conditions:

$$B(z, \omega) \leqslant K(\cos\theta)^2 \sin\theta\, |z|^3, \qquad (z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2 \tag{1.6}$$

$$A^* := \sup_{z \in \mathbf{R}^3} \frac{1}{1+|z|} \int_{\mathbf{S}^2} B(z, \omega)\, d\omega < \infty \tag{1.7}$$

and their special case:

$$B(z, \omega) = \min\{K(\cos\theta)^2 \sin\theta\, |z|^3, b(\theta)|z|^{\beta}\}, \quad (z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2, \quad 0 \leqslant \beta \leqslant 1 \tag{1.8}$$

$$A := 4\pi \int_0^{\pi/2} b(\theta) \sin\theta\, d\theta < \infty \tag{1.9}$$

where K is a positive constant, and, in (1.6) and (1.8), $\theta = \arccos(|z|^{-1} |\langle z, \omega \rangle|)$ (for $|z| = 0$ we define $\theta = 0$), and the angular function $b(\theta)$ is continuous and strictly positive in the open interval $(0, \pi/2)$.

Note that although the high nonlinearity of Eq. (BBE) can be reduced through the following identity

$$f'f'_*(1+\varepsilon f)(1+\varepsilon f_*) - ff_*(1+\varepsilon f')(1+\varepsilon f'_*)$$
$$= f'f'_*(1+\varepsilon f+\varepsilon f_*) - ff_*(1+\varepsilon f'+\varepsilon f'_*) \tag{1.10}$$

the "Bose parts" i.e., the collision integrals

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\{f'f'_*f,\ f'f'_*f_*,\ ff_*f',\ ff_*f'_*\}\,d\omega\,dv_* \tag{1.11}$$

are still difficult to deal with because in general the set $\{f(\cdot,t)\}_{t\geqslant 0}$ is not L^1-weakly compact.

In Section 2 we prove that the cutoff condition (1.6) insures the L^1-boundness of the "Bose parts" for isotropic functions in $L^1(\mathbf{R}^3)$. In Sections 3 and 4 we use this L^1-boundness to prove the global existence of isotropic solutions, entropy identity (1.1), and a moment production estimates of Wennberg's type (ref. 18) which implies the uniqueness of conservative solutions of Eq. (BBE). Here and below the *conservative solution* means the solution that conserves the mass, momentum, and energy, i.e., $\int_{\mathbf{R}^3} f(v,t)\{1, v, |v|^2\}\,dv$ are constants in $t\in[0,\infty)$. In Sections 5 and 6 we study the long-time behavior of isotropic solutions of Eq. (BBE), where the velocity concentration at the very low temperature $T<T_b$ are proven, and, thanks to the moment estimates, some very high temperature conditions for weakly converging to equilibrium states are also given.

Throughout this paper, whenever saying that a function set is weakly compact in L^1, it always means that it is weakly sequentially compact in L^1 (Dunford and Schwartz[9]). Isotropic (or radial) functions in v used in this paper are often denoted as

$$f(v)=\bar{f}(|v|), \qquad f(v,t)=\bar{f}(|v|,t), \text{ etc.}$$

Notations f, f_*, f' and f'_* appeared in collision integrals are the same as usual, i.e.,

$$f=f(v,\cdot), \quad f_*=f(v_*,\cdot), \quad f'=f(v',\cdot), \quad f'_*=f(v'_*,\cdot)$$

where v, v_* and v', v'_* are velocities of two particles before and after their collision respectively, and they have the following relations which have been frequently used in the change of integral variables:

$$v' = v - \langle v - v_*, \omega \rangle \, \omega, \qquad v'_* = v_* + \langle v - v_*, \omega \rangle \, \omega, \qquad \omega \in \mathbf{S}^2$$

$$v' + v'_* = v + v_* \text{AAB}, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2,$$

$$|\langle v' - v'_*, \omega \rangle| = |\langle v - v_*, \omega \rangle|, \qquad |v' - v'_*| = |v - v_*|$$

2. SOME PROPERTIES OF COLLISION INTEGRALS

In this section, under the cutoff conditions (1.6) and (1.7), we prove the L^1-boundness and the weak convergence of the "Bose parts" (1.11). Let

$$\mathbf{R}^2(z) := \{ x \in \mathbf{R}^3 \mid x \perp z \}, \qquad \mathbf{S}^1(z) := \{ \omega \in \mathbf{S}^2 \mid \omega \perp z \}, \qquad z \in \mathbf{R}^3 \backslash \{0\}$$

It is obvious that

$$\mathbf{R}^2(\pm \lambda z) = \mathbf{R}^2(z), \qquad \mathbf{S}^1(\pm \lambda z) = \mathbf{S}^1(z), \quad \forall \lambda > 0, \quad \forall z \in \mathbf{R}^3 \backslash \{0\} \tag{2.1}$$

Let $d^{\perp}\omega$, $d^{\perp}x$ denote the Lebesgue measures on the circle $\mathbf{S}^1(z)$ and on the plan $\mathbf{R}^2(z)$ respectively, i.e., (for instance) for $\varphi \in C(\mathbf{S}^2)$, $0 \leqslant \psi \in C(\mathbf{R}^3 \backslash \{0\})$

$$\int_{\mathbf{S}^1(z)} \varphi(\omega) \, d^{\perp}\omega := \int_0^{2\pi} \varphi(\cos(\theta) \, \mathrm{i} + \sin(\theta) \, \mathrm{j}) \, d\theta, \qquad \mathrm{i}, \, \mathrm{j} \in \mathbf{S}^1(z), \; \mathrm{i} \perp \mathrm{j},$$

$$\int_{\mathbf{R}^2(z)} \psi(x) \, d^{\perp}x = \int_0^{\infty} r \int_{\mathbf{S}^1(z)} \psi(r\omega) \, d^{\perp}\omega \, dr, \qquad z \in \mathbf{R}^3 \backslash \{0\}$$

Lamma 1. Let $F(x, y) \geqslant 0$ be continuous on $(\mathbf{R}^3 \backslash \{0\}) \times (\mathbf{R}^3 \backslash \{0\})$, $a(r, \rho) \geqslant 0$ be measurable on $(0, \infty) \times (0, \infty)$. Then

$$\int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} F(x, y) \, a(|x|, |y|) \, d^{\perp}y$$
$$= \int_{\mathbf{R}^3} dy \int_{\mathbf{R}^2(y)} \frac{|x|}{|y|} F(x, y) \, a(|x|, |y|) \, d^{\perp}x$$

Proof. This is a consequence of the following equality (Lu[13])

$$\int_{\mathbf{S}^2} \left[\int_{\mathbf{S}^1(\sigma)} G(\sigma, \omega) \, d^{\perp}\omega \right] d\sigma = \int_{\mathbf{S}^2} \left[\int_{\mathbf{S}^1(\omega)} G(\sigma, \omega) \, d^{\perp}\sigma \right] d\omega, \quad \forall G \in C(\mathbf{S}^2 \times \mathbf{S}^2) \tag{2.2}$$

Using (2.1), (2.2) and Fubini-Tonelli's theorem we have

$$\int_{\mathbf{R}^3} dy \int_{\mathbf{R}^2(y)} \frac{|x|}{|y|} F(x, y)\, a(|x|, |y|)\, d^{\perp}x$$

$$= \int_0^{\infty} \rho^2 \int_{\mathbf{S}^2} \left(\int_0^{\infty} r \int_{\mathbf{S}^1(\sigma)} \frac{r}{\rho} F(r\omega, \rho\sigma)\, a(r, \rho)\, d^{\perp}\omega\, dr \right) d\sigma\, d\rho$$

$$= \int_0^{\infty} \int_0^{\infty} \rho r^2 \left[\int_{\mathbf{S}^2} \left(\int_{\mathbf{S}^1(\sigma)} F(r\omega, \rho\sigma)\, d^{\perp}\omega \right) d\sigma \right] a(r, \rho)\, dr\, d\rho$$

$$= \int_0^{\infty} \int_0^{\infty} \rho r^2 \left[\int_{\mathbf{S}^2} \left(\int_{\mathbf{S}^1(\omega)} F(r\omega, \rho\sigma)\, d^{\perp}\sigma \right) d\omega \right] a(r, \rho)\, dr\, d\rho$$

$$= \int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} F(x, y)\, a(|x|, |y|)\, d^{\perp}y \quad \blacksquare$$

The following lemma is the Carleman's representation.[4]

Lemma 2. Let F be a nonnegative measurable function on $\mathbf{R}^3 \times \mathbf{R}^3$ such that for any fixed v and $v_* \in \mathbf{R}^3$, $F(v, \cdot)$ and $F(\cdot, v_*)$ are measurable on $\mathbf{R}^3$. Then for all $v \in \mathbf{R}^3$

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, F(v', v'_*)\, d\omega\, dv_*$$

$$= 2 \int_{\mathbf{R}^3} \frac{1}{|x|^2} \left[\int_{\mathbf{R}^2(x)} \bar{B}\left(|x - y|, \frac{|x|}{|x - y|} \right) F(v - x, v - y)\, d^{\perp}y \right] dx \qquad (2.3)$$

Lemma 3. If $x, y, v \in \mathbf{R}^3$ and $x \perp y$, then $|x|\, |y| \leqslant |x|\, |v - y| + |y|\, |v - x|$.

Proof. Let $\mathrm{i} = x/|x|$, $\mathrm{j} = y/|y|$, $\lambda = 1/|x|$ $\mu = 1/|y|$. Then, since $\mathrm{i} \perp \mathrm{j}$, we have

$$\begin{aligned}
[(|x|\, |v - y| + |y|\, |v - x|)/(|x|\, |y|)]^2 &= (|\lambda v - \mathrm{i}| + |\mu v - \mathrm{j}|)^2 \\
&\geqslant |\lambda v - \mathrm{i}|^2 + |\mu v - \mathrm{j}|^2 \\
&\geqslant 1 + (1 - |v|\, |\lambda \mathrm{i} + \mu \mathrm{j}|)^2 \geqslant 1 \quad \blacksquare
\end{aligned}$$

The following proposition implies the L^1-boundness of the "Bose parts" (1.11).

Proposition 2. Suppose the kernel $B(z, \omega)$ satisfies (1.6). Then there is a measurable function $\hat{B}$ on $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$ defined through B, having the following three properties:

(1) $0 \leqslant \hat{B}(v, x, y) \leqslant 2K$, $(v, x, y) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$.

(2) For any sequence of kernels $B_n(z, \omega)$ satisfying $0 \leqslant B_n(z, \omega) \leqslant B(z, \omega)$ and $\lim_{n \to \infty} B_n(z, \omega) = B(z, \omega)$ for all $(z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2$,

$$\lim_{n \to \infty} \widehat{B_n}(v, x, y) = \hat{B}(v, x, y), \qquad (v, x, y) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$$

(3) For any $\Phi(|\cdot|, |\cdot|) \in L^1(\mathbf{R}^3 \times \mathbf{R}^3)$,

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, \Phi(|v'|, |v'_*|)\, d\omega\, dv_*$$
$$= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \hat{B}(v, x, y)\, \Phi(|x|, |y|)\, 1_{\{|x|^2 + |y|^2 > |v|^2\}}\, dx\, dy, \qquad \text{a.e. } v \in \mathbf{R}^3 \tag{2.4}$$

In particular, for any isotropic functions f, g and h which are all in $L^1(\mathbf{R}^3)$, and for any $\phi \in L^\infty(\mathbf{R}^3_+)$,

$$\iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f(v)\, g(v')\, h(v'_*)\, \phi(|v|, |v'|, |v'_*|)\, d\omega\, dv_*\, dv$$
$$= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3} \hat{B}(v, x, y)\, f(v)\, g(x)\, h(y)\, \phi(|v|, |x|, |y|)$$
$$\times 1_{\{|x|^2 + |y|^2 > |v|^2\}}\, dy\, dx\, dv$$

Proof. We may assume that Φ is nonnegative since this implies general case of (2.4). Further, we may assume that Φ is also continuous. In fact, if (2.4) holds for nonnegative continuous functions, then for a general nonnegative function $\Phi(|\cdot|, |\cdot|) \in L^1(\mathbf{R}^3 \times \mathbf{R}^3)$, choose nonnegative continuous functions $\Phi_n(|\cdot|, |\cdot|) \in L^1(\mathbf{R}^3 \times \mathbf{R}^3)$ such that Φ_n converge in $L^1(\mathbf{R}^3 \times \mathbf{R}^3)$ to Φ. By Fatou's Lemma, we see that Φ satisfies the inequality for (2.4), i.e., the left hand side of (2.4) $\leqslant$ the right hand side of (2.4). Thus, the nonnegative functions $|\Phi - \Phi_n|$ also satisfy the same inequality and therefore (2.4) follows from L^1-convergence.

For $v \in \mathbf{R}^3$ and $x, y \in \mathbf{R}^3 \backslash \{0\}$ with $x \perp y$, define

$$B^{\#}(v, x, y) = \frac{\bar{B}\left(|x - y|, \dfrac{|x|}{|x - y|}\right)}{|x|\,(|x|\,|v - y| + |y|\,|v - x|)}$$

and $B^{\#}(v, x, y)=0$, otherwise. Since, by assumption on B (choose $z=x-y$, $\omega=x/|x|$),

$$\bar{B}\left(|x-y|, \frac{|x|}{|x-y|}\right) \leqslant K\left(\frac{|x|}{|x-y|}\right)^2 \frac{|y|}{|x-y|}\, |x-y|^3 \\ = K\, |x|^2\, |y|$$

for all $x, y \in \mathbf{R}^3 \setminus \{0\}$ with $x \perp y$, it follows from Lemma 3 that

$$B^{\#}(v, x, y) \leqslant K, \qquad (v, x, y) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \tag{2.5}$$

Let $\mathscr{I}(\Phi)(v)$ be the left hand side of (2.4). Then using Carleman's representation (2.3), Lemma 1 and $|x-y|=\sqrt{|x|^2+|y|^2}$ for $x \perp y$, we have

$$\begin{aligned}
\mathscr{I}(\Phi)(v) &= 2\int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} \frac{\Phi(|v-x|, |v-y|)}{|x|^2}\, \bar{B}\left(|x-y|, \frac{|x|}{|x-y|}\right) d^{\perp}y \\
&= 2\int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} \frac{\Phi(|v-x|, |v-y|)}{|x|} \\
&\quad \times (|x|\, |v-y| + |y|\, |v-x|)\, B^{\#}(v, x, y)\, d^{\perp}y \\
&= 2\int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} \Phi(|v-x|, |v-y|)\, |v-y|\, B^{\#}(v, x, y)\, d^{\perp}y \\
&\quad + 2\int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} \Phi(|v-x|, |v-y|)\, |v-x|\, \frac{|y|}{|x|}\, B^{\#}(v, x, y)\, d^{\perp}y \\
&= 2\int_{\mathbf{R}^3} dx \int_{\mathbf{R}^2(x)} \Phi(|v-x|, |v-y|)\, |v-y|\, B^{\#}(v, x, y)\, d^{\perp}y \\
&\quad + 2\int_{\mathbf{R}^3} dy \int_{\mathbf{R}^2(y)} \Phi(|v-x|, |v-y|)\, |v-x|\, B^{\#}(v, x, y)\, d^{\perp}x \\
&=: \mathscr{I}_1(\Phi)(v) + \mathscr{I}_2(\Phi)(v)
\end{aligned}$$

Now let $v_x = \langle v, x/|x| \rangle (x/|x|)$, $\tilde{v}_x = v - v_x$. Then $\tilde{v}_x \in \mathbf{R}^2(x)$. Using the translation $y \to y + \tilde{v}_x$ in $\mathbf{R}^2(x)$ we have, for the inner integral in $\mathscr{I}_1(\Phi)(v)$,

$$\int_{\mathbf{R}^2(x)} \Phi(|v-x|, |v-y|)\, |v-y|\, B^{\#}(v, x, y)\, d^{\perp}y$$

$$= \int_{\mathbf{R}^2(x)} \Phi(|v-x|, |v_x-y|)\, |v_x-y|\, B^{\#}(v, x, y+\tilde{v}_x)\, d^{\perp}y$$

$$= \int_0^{\infty} \rho\Phi(|v-x|, \sqrt{\rho^2+|v_x|^2})\, \sqrt{\rho^2+|v_x|^2}$$

$$\times \left[\int_{\mathbf{S}^1(x)} B^{\#}(v, x, \rho\omega+\tilde{v}_x)\, d^{\perp}\omega\right] d\rho$$

$$= \int_{|v_x|}^{\infty} r^2\Phi(|v-x|, r) \left[\int_{\mathbf{S}^1(x)} B^{\#}(v, x, \sqrt{r^2-|v_x|^2}\,\omega+\tilde{v}_x)\, d^{\perp}\omega\right] dr$$

$$= \frac{1}{4\pi}\int_{\mathbf{R}^3} \Phi(|v-x|, |y|)\, 1_{\{|y|>|v_x|\}}$$

$$\times \left[\int_{\mathbf{S}^1(x)} B^{\#}(\, x, \sqrt{|y|^2-|v_x|^2}\,\omega+\tilde{v}_x)\, d^{\perp}\omega\right] dy$$

If we define

$$\hat{B}_1(v, x, y) = 1_{\{|y|>|v_x|\}}\, 1_{\{x\neq 0\}}\, \frac{1}{2\pi}\int_{\mathbf{S}^1(x)} B^{\#}(v, x, \sqrt{|y|^2-|v_x|^2}\,\omega+\tilde{v}_x)\, d^{\perp}\omega$$

then

$$\mathscr{I}_1(\Phi)(v) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \Phi(|v-x|, |y|)\, \hat{B}_1(v, x, y)\, 1_{\{|v-x|^2+|y|^2>|v|^2\}}\, dx\, dy$$

where the factor $1_{\{|v-x|^2+|y|^2>|v|^2\}}$ comes from the fact that $|y|>|v_x|$ implies $|v-x|^2+|y|^2>|v|^2$. Similarly we have

$$\mathscr{I}_2(\Phi)(v) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \Phi(|x|, |v-y|)\, \hat{B}_2(v, x, y)\, 1_{\{|x|^2+|v-y|^2>|v|^2\}}\, dx\, dy$$

with

$$\hat{B}_2(v, x, y) = 1_{\{|x|>|v_y|\}}\, 1_{\{y\neq 0\}}\, \frac{1}{2\pi}\int_{\mathbf{S}^1(y)} B^{\#}(v, \sqrt{|x|^2-|v_y|^2}\,\omega+\tilde{v}_y, y)\, d^{\perp}\omega$$

Therefore (2.4) follows by taking

$$\hat{B}(v, x, y) = \hat{B}_1(v, v-x, y) + \hat{B}_2(v, x, v-y)$$

which is bounded by $2K$ (using (2.5)) and possesses the second property. ∎

Let $\{f_n\}_{n=1}^{\infty}$ be weakly compact in $L^1(\mathbf{R}^k)$, and let $\{g_n\}_{n=1}^{\infty} \subset L^{\infty}(\mathbf{R}^k)$ satisy $\sup_{n \geqslant 1} \|g_n\|_{L^{\infty}(\mathbf{R}^k)} < \infty$ and $\lim_{n \to \infty} g_n(x) = 0$ for a.e. $x \in \mathbf{R}^k$. It is easily shown that $\lim_{n \to \infty} \|f_n g_n\|_{L^1(\mathbf{R}^k)} = 0$. By induction, this property implies the following:

Lemma 4. Let $d = d_1 + d_2 + \cdots + d_N, f_j^n, f_j \in L^1(\mathbf{R}^{d_j})$ $(j = 1, 2,..., N)$, ψ_n, $\psi \in L^{\infty}(\mathbf{R}^d)$ $(n = 1, 2,...)$ satisfy $f_j^n \rightharpoonup f_j$ $(n \to \infty)$ weakly in $L^1(\mathbf{R}^{d_j})$ $(j = 1, 2,..., N)$, $\sup_{n \geqslant 1} \|\psi_n\|_{L^{\infty}(\mathbf{R}^d)} < \infty$, and

$$\lim_{n \to \infty} \psi_n(x) = \psi(x), \qquad \text{a.e.} \quad x = (x_1, x_2,..., x_N) \in \mathbf{R}^d$$

Then

$$\lim_{n \to \infty} \int_{\mathbf{R}^d} f_1^n(x_1)\, f_2^n(x_2) \cdots f_N^n(x_N)\, \psi_n(x)\, dx$$

$$= \int_{\mathbf{R}^d} f_1(x_1)\, f_2(x_2) \cdots f_N(x_N)\, \psi(x)\, dx \quad ∎$$

For $s \geqslant 0$, introduce a subclass of $L^1(\mathbf{R}^3)$:

$$L_s^1(\mathbf{R}^3) = \{f \colon \mathbf{R}^3 \to \mathbf{R} \mid f(v)(1+|v|^2)^{s/2} \in L^1(\mathbf{R}^3)\}$$

with the norm

$$\|f\|_{L_0^1} = \|f\|_{L^1} = \|f\|_{L^1(\mathbf{R}^3)}, \quad \|f\|_{L_s^1} = \|f\|_{L_s^1(\mathbf{R}^3)} = \int_{\mathbf{R}^3} |f(v)|\, (1+|v|^2)^{s/2}\, dv$$

Proposition 3. Suppose the kernel $B(z, \omega)$ satisfy (1.6)–(1.7), and suppose the kernels $B_n(z, \omega)$ satisfy

$$0 \leqslant B_n(x, \omega) \leqslant B(z, \omega), \qquad \lim_{n \to \infty} B_n(z, \omega) = B(z, \omega), \qquad (z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2$$

Let f_n, f be nonnegative isotropic functions in $L_2^1(\mathbf{R}^3)$ satisfying $\sup_{n \geqslant 1} \|f_n\|_{L_2^1} < \infty$ and $f_n \rightharpoonup f$ $(n \to \infty)$ weakly in $L^1(\mathbf{R}^3)$. Then for any bounded measurable function ϕ on $\mathbf{R}_+^3$,

$$\lim_{n\to\infty}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v-v_*,\omega)$$
$$\times f'_n f'_{n*}(1+\varepsilon f_n+\varepsilon f_{n*})\,\phi(|v|,|v'|,|v'_*|)\,d\omega\,dv_*\,dv$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)$$
$$\times f'f'_*(1+\varepsilon f+\varepsilon f_*)\,\phi(|v|,|v'|,|v'_*|)\,d\omega\,dv_*\,dv \tag{2.6}$$

and

$$\lim_{n\to\infty}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v-v_*,\omega)$$
$$\times f_n f_{n*}(1+\varepsilon f'_n+\varepsilon f'_{n*})\,\phi(|v|,|v'|,|v'_*|)\,d\omega\,dv_*\,dv$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\,ff_*(1+\varepsilon f'+\varepsilon f'_*)\,\phi(|v|,|v'|,|v'_*|)\,d\omega\,dv_*\,dv \tag{2.7}$$

Proof. By changing variable $(v, v_*)\to(v', v'_*)$ and noting that $|v_*|$ is a function of $(|v|, |v'|, |v'_*|)$, we see that (2.7) is equivalent to (2.6). Thus, we need only to prove (2.6). Since $\sup_{n\geqslant 1}\|f_n\|_{L^1_2(\mathbf{R}^3)}<\infty$, it is easily seen that

$$f_n(v)(1+|v|^2)^{1/2} \rightharpoonup f(v)(1+|v|^2)^{1/2} \qquad (n\to\infty) \text{ weakly in } L^1(\mathbf{R}^3)$$

By Proposition 2 we have

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v-v_*,\omega)\,f'_n f'_{n*}(1+\varepsilon f_n+\varepsilon f_{n*})\,\phi(|v|,|v'|,|v'_*|)\,d\omega\,dv_*\,dv$$
$$=\iint_{\mathbf{R}^3\times\mathbf{R}^3} f_n(v)(1+|v|^2)^{1/2} f_n(v_*)(1+|v_*|^2)^{1/2}\,I_1(B_n,\phi)(v,v_*)\,dv_*\,dv$$
$$+\varepsilon\sum_{j=2}^{3}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} f_n(v)\,f_n(x)\,f_n(y)\,I_j(B_n,\phi)(v,x,y)\,dx\,dy\,dv$$

where

$$I_1(B,\phi)(v,v_*)=\frac{1}{(1+|v|^2)^{1/2}\,(1+|v_*|^2)^{1/2}}$$
$$\times\int_{\mathbf{S}^2} B(v-v_*,\omega)\,\phi(|v'|,|v|,|v_*|)\,d\omega$$

$$I_2(B,\phi)(v,x,y)=\hat{B}(v,x,y)\,\phi(|v|,|x|,|y|)\,1_{\{|x|^2+|y|^2>|v|^2\}}$$
$$I_3(B,\phi)(v,x,y)=\hat{B}(v,x,y)\,\phi(\sqrt{|x|^2+|y|^2-|v|^2},|y|,|x|)\,1_{\{|x|^2+|y|^2>|v|^2\}}$$

By assumption and Proposition 2 we have

$$\sup_{n\geqslant 1} \|I_1(B_n, \phi)\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)} \leqslant 2A^* \|\phi\|_{L^\infty(\mathbf{R}^3_+)} < \infty,$$

$$\lim_{n\to\infty} I_1(B_n, \phi)(v, v_*) = I_1(B, \phi)(v, v_*) \qquad \text{a.e.} \quad (v, v_*) \in \mathbf{R}^3\times\mathbf{R}^3$$

and

$$\sup_{n\geqslant 1} \|I_j(B_n, \phi)\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3)} \leqslant 2K \|\phi\|_{L^\infty(\mathbf{R}^3_+)} < \infty$$

$$\lim_{n\to\infty} I_j(B_n, \phi)(v, x, y)$$

$$= I_j(B, \phi)(v, x, y) \qquad \text{a.e.} \quad (v, x, y) \in \mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3, \qquad j = 2, 3$$

Thus applying Lemma 4 we have

$$\lim_{n\to\infty} \iint_{\mathbf{R}^3\times\mathbf{R}^3} f_n(v)(1+|v|^2)^{1/2} f_n(v_*)(1+|v_*|^2)^{1/2} I_1(B_n, \phi)(v, v_*)\, dv_*\, dv$$

$$= \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v)(1+|v|^2)^{1/2} f(v_*)(1+|v_*|^2)^{1/2} I_1(B, \phi)(v, v_*)\, dv_*\, dv$$

$$= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)\, f'f'_* \phi(|v|, |v'|, |v'_*|)\, d\omega\, dv_*\, dv$$

and

$$\lim_{n\to\infty} \sum_{j=2}^{3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} f_n(v)\, f_n(x)\, f_n(y)\, I_j(B_n, \phi)(v, x, y)\, dx\, dy\, dv$$

$$= \sum_{j=2}^{3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} f(v)\, f(x)\, f(y)\, I_j(B, \phi)(v, x, y)\, dx\, dy\, dv$$

$$= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)\, f'f'_*(f+f_*)\, \phi(|v|, |v'|, |v'_*|)\, d\omega\, dv_*\, dv$$

These prove (2.6). ∎

3. EXISTENCE(I), CONSERVATION OF ENERGY, AND ENTROPY IDENTITY

Let $Q(f, f)$ be the collision integral in Eq. (BBE), i.e.,

$$Q(f, f)(v, t) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times [f'f'_*(1 + \varepsilon f)(1 + \varepsilon f_*) - ff_*(1 + \varepsilon f')(1 + \varepsilon f'_*)] \, d\omega \, dv_*$$

Because of the reduction identity (1.10), we define

$$Q^+(f, f)(v, t) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \; f'f'_*(1 + \varepsilon f + \varepsilon f_*) \, d\omega \, dv_*$$

$$Q^-(f, f)(v, t) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \; ff_*(1 + \varepsilon f' + \varepsilon f'_*) \, d\omega \, dv_*$$

If $Q^{\pm}(f, f)(v, t)$ are finite, the decomposition

$$Q(f, f)(v, t) = Q^+(f, f)(v, t) - Q^-(f, f)(v, t)$$

is meaningful.

Solutions of Eq. (BBE). Given an initial datum $0 \leqslant f_0 \in L^1(\mathbf{R}^3)$. We say that a function f is a (global) mild solution of Eq. (BBE) on $\mathbf{R}^3 \times [0, \infty)$ with $f|_{t=0} = f_0$ if f is nonnegative and satisfies the following

(i) f is measurable on $\mathbf{R}^3 \times [0, \infty)$, and $f \in L^\infty([0, \infty); L^1(\mathbf{R}^3))$.

(ii)

$$\int_0^t Q^{\pm}(f, f)(v, \tau) \, d\tau < \infty, \qquad v \in \mathbf{R}^3 \backslash Z, \quad t \in [0, \infty),$$

$$f(v, t) = f_0(v) + \int_0^t Q(f, f)(v, \tau) \, d\tau, \qquad v \in \mathbf{R}^3 \backslash Z, \quad t \in [0, \infty) \tag{3.1}$$

where Z is a null set independent of t.

Note that if instead of (ii), f satisfies $Q^{\pm}(f, f) \in L^1(\mathbf{R}^3 \times [0, t_1])$, $\forall t_1 > 0$, and

$$f(v, t) = f_0(v) + \int_0^t Q(f, f)(v, \tau) \, d\tau, \quad t \in [0, \infty), \; v \in \mathbf{R}^3 \backslash Z_t, \; \text{meas}(Z_t) = 0$$

then f can be modified on v-null sets such that the modification of f satisfies (ii). In fact if we define $\tilde{f}(v,t) := |f_0(v) + \int_0^t Q(f,f)(v,\tau)\,d\tau|$ then since $f \geqslant 0$ is measurable on $\mathbf{R}^3 \times [0,\infty)$, we have $\tilde{f}(v,t) = f(v,t)$, $t \in [0,\infty)$, $v \in \mathbf{R}^3 \setminus Z_t$, $Q^{\pm}(\tilde{f},\tilde{f}) \in L^1(\mathbf{R}^3 \times [0,t_1])$, $\forall t_1 > 0$, and

$$\tilde{f}(v,t) = f_0(v) + \int_0^t Q(\tilde{f},\tilde{f})(v,\tau)\,d\tau, \quad t \in [0,\infty), \quad v \in \mathbf{R}^3 \setminus \tilde{Z}_t, \quad \operatorname{meas}(\tilde{Z}_t) = 0$$

Since both $\tilde{f}(v,t)$ and $\int_0^t Q^{\pm}(\tilde{f},\tilde{f})(v,\tau)\,d\tau$ are continuous with respect to t for a.e. $v \in \mathbf{R}^3$, it follows from Lemma 5 (see below) that $\tilde{f}$ satisfies (ii), and so $\tilde{f}$ is a mild solution of Eq. (BBE). In this sense, we do not distinguish between f and its modifications on v-null sets. In this paper, a function f is a solution of Eq. (BBE) always means f is a mild solution of Eq. (BBE).

The following lemma mentioned above is an easy application of Fubini's theorem.

Lemma 5. Let $I \subset \mathbf{R}$ be an interval, and let f_1, f_2 be measurable functions on $\mathbf{R}^N \times I$ satisfying

(i) $\exists$ null sets $Z_i \subset \mathbf{R}^N$ such that $\forall v \in \mathbf{R}^N \setminus Z_i$, $t \mapsto f_i(v,t)$ is continuous on I, $(i = 1, 2)$.

(ii) $f_1(v,t) = f_2(v,t)$ for a.e. $(v,t) \in \mathbf{R}^N \times I$.

Then there exists a null set $Z \subset \mathbf{R}^N$, which is independent of t, such that

$$f_1(v,t) = f_2(v,t), \qquad \forall t \in I, \quad \forall v \in \mathbf{R}^N \setminus Z \quad \blacksquare$$

Before proving the existence of isotropic solutions of Eq. (BBE), let us first recall an important fact: If $0 \leqslant f \in L^1_2(\mathbf{R}^3)$ is an isotropic function, then $Q^{\pm}(f,f)$ and therefore $Q(f,f)$ are also isotropic functions because the kernel $B(v - v_*, \omega)$ depends only on $|v - v_*|$ and $|\langle v - v_*, \omega \rangle|$.

Theorem 1. Suppose the kernel $B(z,\omega)$ satisfies (1.6) and $\int_{\mathbf{S}^2} B(z,\omega)\,d\omega$ is bounded on $\mathbf{R}^3$. Then for any isotropic initial datum $0 \leqslant f_0 \in L^1_2(\mathbf{R}^3)$, there exists a unique conservative isotropic solution f of Eq. (BBE) in $C^1([0,\infty), L^1_2(\mathbf{R}^3))$ with $f|_{t=0} = f_0$. Moreover if for some $s > 2$, $f_0 \in L^1_s(\mathbf{R}^3)$, then

$$\|f(\cdot,t)\|_{L^1_s} \leqslant \|f_0\|_{L^1_s}\, e^{at}, \qquad t \geqslant 0 \tag{3.2}$$

where $a = 2^{s+1} A^* \|f_0\|_{L^1_2} + 3 \cdot 2^{s/2} \varepsilon K \|f_0\|^2_{L^1_2}$, A^* is given by (1.7).

Proof. For $\delta > 0$, let $\mathscr{B}_\delta$ be the collection of isotropic functions $f(v, t) = \bar{f}(|v|, t)$ satisfying (i) $\bar{f}(r, t)$ are measurable on $[0, \infty) \times [0, \delta]$, (ii) for any $t \in [0, \delta]$, $r \mapsto \bar{f}(r, t)$ are measurable on $[0, \infty)$, and (iii)

$$\|f\|_\delta := \sup_{t \in [0, \delta]} \|f(\cdot, t)\|_{L^1_2} \leqslant 2 \|f_0\|_{L^1_2}$$

Let $C_0 = \sup_{z \in \mathbf{R}^3} \int_{\mathbf{S}^2} B(z, \omega)\, d\omega$. We first prove that there exists a $\delta > 0$, which depends only on the constants C_0, εK and $\|f_0\|_{L^1_2}$, such that the existence conclusion in the theorem holds on the time interval $[0, \delta]$. Let $f, g \in \mathscr{B}_\delta$. By Proposition 2 we have

$$\begin{aligned}
&\int_{\mathbf{R}^3} Q^{\pm}(|f|, |f|)(v, t)(1 + |v|^2)\, dv \\
&\quad \leqslant \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, |f|\, (1 + |v|^2)\, |f_*|\, (1 + |v_*|^2)\, d\omega\, dv_*\, dv \\
&\qquad + 2\varepsilon \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, |f|\, (1 + |v|^2)\, |f'| \\
&\qquad \times (1 + |v'|^2)\, |f'_*|\, (1 + |v'_*|^2)\, d\omega\, dv_*\, dv \\
&\quad \leqslant C_0 (\|f(\cdot, t)\|_{L^1_2})^2 + 4\varepsilon K (\|f(\cdot, t)\|_{L^1_2})^3 \\
&\leqslant [4C_0 \|f_0\|_{L^1_2} + 32\varepsilon K (\|f_0\|_{L^1_2})^2]\, \|f_0\|_{L^1_2} =: C_0(f_0)\, \|f_0\|_{L^1_2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^3} |Q(|f|, |f|)(v, t) - Q(|g|, |g|)(v, t)|\, (1 + |v|^2)\, dv \\
&\quad \leqslant 2 \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \\
&\qquad \times \big| |f'|\, |f'_*|\, (1 + \varepsilon |f| + \varepsilon |f_*|) - |g'|\, |g'_*|\, (1 + \varepsilon |g| + \varepsilon |g_*|) \big| \\
&\qquad \times (1 + |v|^2 + |v_*|^2)\, d\omega\, dv_*\, dv \\
&\quad \leqslant 2C_0 \|f - g\|_\delta (\|f\|_\delta + \|g\|_\delta) + 8\varepsilon K \|f - g\|_\delta (\|f\|_\delta + \|g\|_\delta)^2 \\
&\quad \leqslant [8C_0 \|f_0\|_{L^1_2} + 128\varepsilon K (\|f_0\|_{L^1_2})^2]\, \|f - g\|_\delta =: \tilde{C}_0(f_0)\, \|f - g\|_\delta
\end{aligned}$$

If we define

$$\mathscr{J}(f)(v, t) = f_0(v) + \int_0^t Q(|f|, |f|)(v, \tau)\, d\tau, \qquad (v, t) \in \mathbf{R}^3 \times [0, \delta]$$

then

$$\|\mathscr{J}(f)\|_\delta \leqslant \|f_0\|_{L^1_2} + 2C_0(f_0)\,\|f_0\|_{L^1_2}\,\delta$$

and

$$\|\mathscr{J}(f) - \mathscr{J}(g)\|_\delta \leqslant \tilde{C}_0(f_0)\,\|f-g\|_\delta\,\delta$$

Choose $\delta = 1/(2\tilde{C}_0(f_0))(<1/(2C_0(f_0)))$. Then $\mathscr{J}$ is a contractive mapping from the complete matric space $(\mathscr{B}_\delta, \|\cdot - \cdot\|_\delta)$ into itself and therefore $\mathscr{J}$ has a unique fixed point $f \in \mathscr{B}_\delta$, i.e., $\|f - \mathscr{J}(f)\|_\delta = 0$. After a modification for f on v-null sets and using Fubini's theorem, there is a null set $Z_\delta \subset \mathbf{R}^3$, which is independent of t, such that

$$\int_0^\delta Q^\pm(|f|, |f|)(v, t)\, dt < \infty, \qquad v \in \mathbf{R}^3 \backslash Z_\delta$$

and

$$f(v, t) = f_0(v) + \int_0^t Q(|f|, |f|)(v, \tau)\, d\tau, \qquad v \in \mathbf{R}^3 \backslash Z_\delta, \quad t \in [0, \delta]$$

It remains to prove the nonnegativity of the fixed point f. For real number y denote $(y)^+ = \max\{y, 0\}$. Then $\forall v \in \mathbf{R}^3 \backslash Z_\delta$,

$$\begin{aligned}(-f(v,t))^+ &= \int_0^t [-Q(|f|, |f|)(v, \tau)]\, 1_{\{f(v,\tau)\leqslant 0\}}\, d\tau \\ &\leqslant \int_0^t Q^-(|f|, |f|)(v, \tau)\, 1_{\{f(v,\tau)\leqslant 0\}}\, d\tau\end{aligned}$$

and so by $|f|\, 1_{\{f \leqslant 0\}} = (-f)^+$, Proposition 2 and $\|f(\cdot, \tau)\|_{L^1} \leqslant 2\,\|f_0\|_{L^1_2}$ we have

$$\begin{aligned}\|(-f(\cdot, t))^+\|_{L^1} &\leqslant \int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, |f| \\ &\quad \times 1_{\{f\leqslant 0\}}\, |f_*|\, (1 + \varepsilon\, |f'| + \varepsilon\, |f'_*|)\, d\omega\, dv_*\, dv \\ &\leqslant [2C_0\,\|f_0\|_{L^1_2} + 16\varepsilon K\,\|f_0\|^2_{L^1_2}] \\ &\quad \times \int_0^t \|(-f(\cdot, \tau))^+\|_{L^1}\, d\tau, \qquad t \in [0, \delta]\end{aligned}$$

Thus by Gronwall inequality, $\|(-f(\cdot,t))^+\|_{L^1}=0$ for all $t\in[0,\delta]$. Equivalently, $\forall t\in[0,\delta]$, $f(v,t)\geqslant 0$ a.e. $v\in\mathbf{R}^3$. Let $\tilde{f}(v,t)=|\mathscr{J}(f)(v,t)|$. Then $\tilde{f}$ is nonnegative and continuous with respect to $t\in[0,\delta]$ for all $v\in\mathbf{R}^3$, and satisfies $\|\tilde{f}-\mathscr{J}(\tilde{f})\|_\delta=0$. Therefore using Lemma 5 we see that $\tilde{f}$ is a local solution of Eq. (BBE) on the interval $[0,\delta]$. Still denote $\tilde{f}$ by f. Then it is easily seen that f is a unique conservative solution of Eq. (BBE) in the class $C^1([0,\delta], L^1_2(\mathbf{R}^3))$.

By conservation of mass and energy, we have $\|f(\cdot,\delta)\|_{L^1_2}=\|f_0\|_{L^1_2}$. Thus with the same $\delta>0$ and replacing the initial datum f_0 by $f(\cdot,\delta)$, $f(\cdot,2\delta)$,..., respectively, the solution f can be inductively extended to all intervals $[\delta,2\delta]$, $[2\delta,3\delta]$,..., and the extended function f is a unique conservative solution of Eq. (BBE) and belongs to the class $C^1([0,\infty), L^1_2(\mathbf{R}^3))$.

Now we prove (3.2). Suppose $f_0\in L^1_s(\mathbf{R}^3)$ $(s>2)$. Let $\phi_n(v)=(1+|v|^2)^{s/2}\wedge n$, $n\geqslant 1$, where $a\wedge b=\min\{a,b\}$. By $|v'|^2+|v'_*|^2=|v|^2+|v_*|^2$ we have

$$\phi_n(v')\leqslant 2^{s/2-1}[\phi_n(v)+\phi_n(v_*)],\qquad \phi_n(v)\leqslant 2^{s/2-1}[\phi_n(v')+\phi_n(v'_*)]$$

These implies by Proposition 2 that

$$\int_0^t d\tau\int_{\mathbf{R}^3} Q^+(f,f)(v,\tau)\,\phi_n(v)\,dv\leqslant C\int_0^t\|f(\cdot,\tau)\,\phi_n\|_{L^1},\qquad t\geqslant 0$$

where the constant C is independent of n. Thus by Gronwall inequality we obtain

$$\|f(\cdot,t)\,\phi_n\|_{L^1}\leqslant\|f_0\phi_n\|_{L^1}\exp\{Ct\},\qquad t\geqslant 0$$

Letting $n\to\infty$ we see that $f\in L^\infty_{loc}([0,\infty); L^1_s(\mathbf{R}^3))$. Therefore using Povzner's inequality

$$\begin{aligned}&(1+|v'|^2)^{s/2}+(1+|v'_*|^2)^{s/2}-(1+|v|^2)^{s/2}-(1+|v_*|^2)^{s/2}\\&\qquad\leqslant 2^s[(1+|v|^2)^{(s-1)/2}(1+|v_*|^2)^{1/2}+(1+|v|^2)^{1/2}(1+|v_*|^2)^{(s-1)/2}]\end{aligned}$$

and inequalities

$$\begin{aligned}\int_{\mathbf{S}^2}B(v-v_*,\omega)\,d\omega&\leqslant A^*(1+|v-v_*|)\\&\leqslant 2A^*(1+|v|^2)^{1/2}(1+|v_*|^2)^{1/2}\end{aligned}\tag{3.3}$$

and

$$(1+|v|^2)^{s/2}\leqslant 2^{s/2-1}[(1+|v'|^2)^{s/2}+(1+|v'_*|^2)^{s/2}]$$

(which is used for "Bose parts" only) we have

$$\|f(\cdot,t)\|_{L^1_s} = \|f_0\|_{L^1_s(\mathbf{R}^3)} + \tfrac{1}{2}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, ff_*[(1+|v'|^2)^{s/2} + (1+|v'_*|^2)^{s/2} - (1+|v|^2)^{s/2} - (1+|v_*|^2)^{s/2}]\, d\omega\, dv_*\, dv$$
$$+ \varepsilon \int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \times [f'f'_*(f+f_*) - ff_*(f'+f'_*)](1+|v|^2)^{s/2}\, d\omega\, dv_*\, dv$$
$$\leqslant \|f_0\|_{L^1_s} + 2^s\, \tfrac{1}{2}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, ff_* \times [(1+|v|^2)^{(s-1)/2}\,(1+|v_*|^2)^{1/2} + (1+|v|^2)^{1/2}\,(1+|v_*|^2)^{(s-1)/2}]\, d\omega\, dv_*\, dv$$
$$+ \varepsilon \int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f'f'_*f(1+|v|^2)^{s/2}\, d\omega\, dv_*\, dv$$
$$+ 2^{s/2-1}\varepsilon \int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f'f'_*f_* \times [(1+|v'|^2)^{s/2} + (1+|v'_*|^2)^{s/2})]\, d\omega\, dv_*\, dv$$
$$\leqslant \|f_0\|_{L^1_s} + (2^{s+1}A^*\, \|f_0\|_{L^1_2} + 3\cdot 2^{s/2}\varepsilon K\, \|f_0\|^2_{L^1_2}) \times \int_0^t \|f(\cdot,\tau)\|_{L^1_s}\, d\tau, \qquad t \geqslant 0$$

Therefore (3.2) follows from Gronwall inequality. ∎

Lemma 6. If $0 \leqslant f \in L^1_2(\mathbf{R}^3)$, then

$$\int_{\mathbf{R}^3} \left| \frac{1}{\varepsilon}(1+\varepsilon f)\log(1+\varepsilon f) - f\,\log f \right| dv \leqslant (2+|\log\varepsilon|)\, \|f\|_{L^1_2} + \frac{1}{\varepsilon}(2\pi)^{3/2}$$

Proof. Using the equality $(1+y)\log(1+y) - y\log y = \log(1+y) + y\log(1+1/y)$, $y \geqslant 0$ we have

$$G(v) := \left| \frac{1}{\varepsilon}(1+\varepsilon f(v))\log(1+\varepsilon f(v)) - f\,\log f(v) \right|$$
$$\leqslant f(v) + f(v)\log(1+(\varepsilon f(v))^{-1}) + f(v)\,|\log\varepsilon|$$

If $\varepsilon f(v) \leqslant e^{-|v|^2}$, the elementary inequality $\log(1+y) \leqslant \sqrt{y}$ $(y \geqslant 0)$ gives $f(v)\log(1+(\varepsilon f(v))^{-1}) \leqslant (1/\varepsilon)\, e^{-(1/2)\,|v|^2}$. Thus

$$G(v) \leqslant (2+|\log \varepsilon|)\, f(v)(1+|v|^2) + \frac{1}{\varepsilon}\, e^{-(1/2)\,|v|^2}$$

This proves the lemma. ∎

Theorem 2. Suppose the kernel B satisfies (1.6)–(1.7). Let $f_0 \geqslant 0$ be an isotropic initial datum in $L^1_2(\mathbf{R}^3)$, and let $f \geqslant 0$ be an isotropic solution of Eq. (BBE) with $f|_{t=0} = f_0$, and $f \in L^\infty([0,\infty); L^1_2(\mathbf{R}^3))$. Then f satisfies the entropy identity (1.1) and the following non-decrease of energy:

$$\int_{\mathbf{R}^3} f(v,s)\,|v|^2\,dv \leqslant \int_{\mathbf{R}^3} f(v,t)\,|v|^2\,dv, \qquad \forall 0 \leqslant s < t < \infty \tag{3.4}$$

In particular, if

$$\int_{\mathbf{R}^3} f(v,t)\,|v|^2\,dv \leqslant \int_{\mathbf{R}^3} f_0(v)\,|v|^2\,dv, \qquad t \geqslant 0$$

then f is a conservative solution.

Proof. The method used in the proof is essentially the same as that in Lu.[14] To completeness, we present it as follows. First of all it is easily proved by Proposition 2 that the solution f conserves the mass. To prove (3.4), consider $\phi_\delta(v) = (1/\delta)\log(1+\delta\,|v|^2)$. By assumption, the kernel B satisfies (3.3). This implies that $Q^\pm(f,f)(v,t)\,\phi_\delta(v) \in L^1(\mathbf{R}^3 \times [0,t_1])$, $\forall t_1 > 0$. Thus, using the integral form (3.1) of the Eq. (BBE) and the equality $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ respectively we have for $0 \leqslant s < t < \infty$

$$\int_{\mathbf{R}^3} f(v,t)\,\phi_\delta(v)\,dv = \int_{\mathbf{R}^3} f(v,s)\,\phi_\delta(v)\,dv + \int_s^t d\tau \int_{\mathbf{R}^3} Q(f,f)(v,\tau)\,\phi_\delta(v)\,dv$$

and

$$\begin{aligned}\int_{\mathbf{R}^3} Q(f,f)(v,\tau)\,\phi_\delta(v)\,dv &= \tfrac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, ff_*(1+\varepsilon f' + \varepsilon f'_*) \\ &\quad \times [\phi_\delta{}' + \phi_{\delta *}' - \phi_\delta - \phi_{\delta *}]\,d\omega\,dv_*\,dv \\ &= \tfrac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, ff_*(1+\varepsilon f' + \varepsilon f'_*) \\ &\quad \times [\psi_\delta(v',v'_*) - \psi_\delta(v,v_*)]\,d\omega\,dv_*\,dv\end{aligned}$$

where

$$\psi_\delta(v, v_*) = \frac{1}{\delta} \log\left(1 + \frac{\delta^2 |v|^2 |v_*|^2}{1 + \delta(|v|^2 + |v_*|^2)}\right)$$

Thus, with

$$I_\delta(f)(v, v_*, \tau) := ff_* \left[\int_{\mathbf{S}^2} B(v - v_*, \omega)(1 + \varepsilon f' + \varepsilon f'_*)\, d\omega \right] \psi_\delta(v, v_*)$$

we obtain

$$\int_{\mathbf{R}^3} f(v, t)\, \phi_\delta(v)\, dv \geqslant \int_{\mathbf{R}^3} f(v, s)\, \phi_\delta(v)\, dv - \tfrac{1}{2} \int_s^t d\tau \iint_{\mathbf{R}^3 \times \mathbf{R}^3} I_\delta(v, v_*, \tau)\, dv_*\, dv \tag{3.5}$$

Next, by inequality $\log(1 + y) \leqslant \sqrt{y}$ $(y \geqslant 0)$ we have

$$\psi_\delta(v, v_*) \leqslant \frac{1}{\delta} \cdot \frac{\delta\, |v|\, |v_*|}{\sqrt{1 + \delta(|v|^2 + |v_*|^2)}} \leqslant |v|\, |v_*|$$

and so by (3.3)

$$\begin{aligned} I_\delta(f)(v, v_*, \tau) &\leqslant 2A^* f(v, \tau)(1 + |v|^2)\, f(v_*, \tau)(1 + |v_*|^2) \\ &\quad + \varepsilon f(v, \tau)\, |v|\, f(v_*, \tau)\, |v_*| \int_{\mathbf{S}^2} B(v - v_*, \omega)(f(v', \tau) \\ &\quad + f(v'_*, \tau))\, d\omega =: F(v, v_*, \tau) \end{aligned}$$

Because $f \in L^\infty([0, \infty), L^1_2(\mathbf{R}^3))$, Proposition 2 implies that the function F belongs to $L^1(\mathbf{R}^3 \times \mathbf{R}^3 \times [0, t_1])$ for all $t_1 > 0$. Since $\psi_\delta(v, v_*) \to 0$ $(\delta \to 0^+)$, it follows from Lebesgue dominated convergence theorem that

$$\int_s^t d\tau \iint_{\mathbf{R}^3 \times \mathbf{R}^3} I_\delta(v, v_*, \tau)\, dv_*\, dv \to 0 \qquad (\delta \to 0^+)$$

Therefore by (3.5), $\phi_\delta(v) \leqslant |v|^2$, and $\phi_\delta(v) \to |v|^2$ $(\delta \to 0^+)$, we obtain (3.4):

$$\int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv \geqslant \int_{\mathbf{R}^3} f(v, s)\, |v|^2\, dv, \qquad \forall 0 \leqslant s < t < \infty$$

Now we prove the entropy identity (1.1). Define for $g(v,t)\geqslant 0$

$$J(g)(v,t)=\frac{1}{\varepsilon}(1+\varepsilon g(v,t))\log(1+\varepsilon g(v,t))-g(v,t)\log g(v,t)$$

For $n\geqslant 1$, let $\Phi(v)=(1+|v|^2)^{-4}$, $\phi_n(v)=(1/n)\,\Phi(v)$ and

$$f_n(v,t)=f(v,t)+\phi_n(v), \qquad f_{n0}(v)=f_0(v)+\phi_n(v)$$

Since $f_n\leqslant f_1$, it is easily shown that $|J(f_n)(v,t)|\leqslant|J(f_1)(v,t)|+2f_1(v,t)\times|\log\varepsilon|$, $n\geqslant 1$. And by Lemma 6 we have

$$\int_{\mathbf{R}^3}[\,|J(f_1)(v,t)|+2f_1(v,t)\,|\log\varepsilon|\,]\,dv$$
$$\leqslant(2+3\,|\log\varepsilon|)\,\|f_1(\cdot,t)\|_{L^1_2}+\frac{1}{\varepsilon}(2\pi)^{3/2}<\infty$$

Since

$$\lim_{n\to\infty}J(f_n)(v,t)=J(f)(v,t), \qquad \forall(v,t)\in\mathbf{R}^3\times[0,\infty)$$

it follows from dominated convergence theorem that

$$\lim_{n\to\infty}S_{f_n}(t)=\lim_{n\to\infty}\int_{\mathbf{R}^3}J(f_n)(v,t)\,dv$$
$$=\int_{\mathbf{R}^3}J(f)(v,t)\,dv=S_f(t), \qquad \forall t\in[0,\infty) \tag{3.6}$$

Next, we note that for a null set $Z\subset\mathbf{R}^3$, $\forall v\in\mathbf{R}^3\backslash Z$, $t\mapsto J(f_n)(v,t)$ is absolutely continuous on $[0,t_1]$ ($\forall t_1>0$). This implies that $\forall v\in\mathbf{R}^3\backslash Z$,

$$J(f_n)(v,t)=J(f_{n0})(v)+\int_0^t Q(f,f)(v,\tau)\log\left(\frac{1+\varepsilon f_n(v,\tau)}{f_n(v,\tau)}\right)d\tau, \quad \forall t\in[0,\infty) \tag{3.7}$$

Moreover, with $C_n=|\log\varepsilon|+8(n/\varepsilon)^{1/8}$, we have

$$\left|\log\left(\frac{1+\varepsilon f_n(v,\tau)}{f_n(v,\tau)}\right)\right|\leqslant C_n(1+|v|^2)^{1/2}$$

Thus by (3.3) and Proposition 2 we have for all $t \in [0, \infty)$,

$$\int_0^t d\tau \int_{\mathbf{R}^3} [Q^+(f, f)(v, \tau) + Q^-(f, f)(v, \tau)] \left| \log\left(\frac{1+\varepsilon f_n}{f_n}\right)\right| dv$$
$$\leqslant \{4C_n A^* [\sup_{\tau \geqslant 0} \|f(\cdot, \tau)\|_{L^1_2}]^2 + 8\varepsilon C_n K [\sup_{\tau \geqslant 0} \|f(\cdot, \tau)\|_{L^1_2}]^3\} \; t < \infty$$

From this integrability and (3.7) we obtain (making use of the change of variables in the collision integrals)

$$S_{f_n}(t) = S_{f_n}(0) - \int_0^t d\tau \int_{\mathbf{R}^3} Q(f, f)(v, \tau) \log F_n(v, \tau)\, dv$$
$$= S_{f_n}(0) + \tfrac{1}{4} \int_0^t d\tau \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)$$
$$\times \Pi_f(v, v_*, \omega, \tau)\, \Gamma_n(F', F'_*, F, F_*)\, d\omega\, dv_*\, dv$$

where

$$F_n = \frac{f_n}{1+\varepsilon f_n}, \qquad F = \frac{f}{1+\varepsilon f}$$
$$\Pi_f(v, v_*, \omega, \tau) = (1+\varepsilon f')(1+\varepsilon f'_*)(1+\varepsilon f)(1+\varepsilon f_*)$$
$$\Gamma_n(F', F'_*, F, F_*) = (F'F'_* - FF_*) \log\left(\frac{F'_n F'_{n*}}{F_n F_{n*}}\right)$$

Let

$$\Gamma_n^+(F', F'_*, F, F_*) = \left[(F'F'_* - FF_*) \log\left(\frac{F'_n F'_{n*}}{F_n F_{n*}}\right)\right]^+$$
$$\Gamma_n^-(F', F'_*, F, F_*) = \left[-(F'F'_* - FF_*) \log\left(\frac{F'_n F'_{n*}}{F_n F_{n*}}\right)\right]^+$$

Then

$$\tfrac{1}{4} \int_0^t d\tau \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, \Pi_f(v, v_*, \omega, \tau)\, \Gamma_n^+(F', F'_*, F, F_*)\, d\omega\, dv_*\, dv$$
$$= S_{f_n}(t) - S_{f_n}(0) + \tfrac{1}{4} \int_0^t d\tau \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)$$
$$\times \Pi_f(v, v_*, \omega, \tau)\, \Gamma_n^-(F', F'_*, F, F_*)\, d\omega\, dv_*\, dv \tag{3.8}$$

Now we prove that, with convergence argument, (3.8) implies the entropy identity (1.1). Write

$$F_n = F + \psi_n \qquad \text{with} \quad \psi_n = \frac{\phi_n}{(1+\varepsilon f_n)(1+\varepsilon f)}$$

Then it can be shown that[13]

$$\begin{aligned}
\Gamma_n^+(F', F'_*, F, F_*) &= \left[(F'F'_* - FF_*) \log\left(\frac{(F'+\psi'_n)(F'_*+\psi'_{n*})}{(F+\psi_n)(F_*+\psi_{n*})} \right) \right]^+ \\
&\leqslant \Gamma(F'F'_*, FF_*) + F'\psi'_{n*} + F'_*\psi'_n \\
&\quad + F\psi_{n*} + F_*\psi_n + \psi'_n\psi'_{n*} + \psi_n\psi_{n*} \\
\Gamma_n^-(F', F'_*, F, F_*) &= \left[-(F'F'_* - FF_*) \log\left(\frac{(F'+\psi'_n)(F'_*+\psi'_{n*})}{(F+\psi_n)(F_*+\psi_{n*})} \right) \right]^+ \\
&\leqslant F'\psi'_{n*} + F'_*\psi'_n + F\psi_{n*} + F_*\psi_n + \psi'_n\psi'_{n*} + \psi_n\psi_{n*}
\end{aligned}$$

where $\Gamma(\cdot, \cdot)$ is given by (1.2). Since $\phi_n = (1/n)\,\Phi$, we have

$$\begin{aligned}
&\Pi_f(v, v_*, \omega, \tau)[F'\psi'_{n*} + F'_*\psi'_n + F\psi_{n*} + F_*\psi_n + \psi'_n\psi'_{n*} + \psi_n\psi_{n*}] \\
&\quad \leqslant \frac{1}{n}\, G(v, v_*, \omega)
\end{aligned}$$

where

$$\begin{aligned}
G(v, v_*, \omega) &= (1+\varepsilon f)(1+\varepsilon f_*)\, f'\Phi'_* + (1+\varepsilon f)(1+\varepsilon f_*)\, f'_*\Phi' \\
&\quad + (1+\varepsilon f')(1+\varepsilon f'_*)\, f\Phi_* + (1+\varepsilon f')(1+\varepsilon f'_*)\, f_*\Phi \\
&\quad + (1+\varepsilon f)(1+\varepsilon f_*)\, \Phi'\Phi'_* + (1+\varepsilon f')(1+\varepsilon f'_*)\, \Phi\Phi_*
\end{aligned}$$

Thus

$$\begin{aligned}
&\Pi_f(v, v_*, \omega, \tau)\, \Gamma_n^+(F', F'_*, F, F_*) \\
&\quad \leqslant \Pi_f(v, v_*, \omega, \tau)\, \Gamma(F'F'_*, FF_*) + \frac{1}{n}\, G(v, v_*, \omega, \tau)
\end{aligned} \tag{3.9}$$

$$\Pi_f(v, v_*, \omega, \tau)\, \Gamma_n^-(F', F'_*, F, F_*) \leqslant \frac{1}{n}\, G(v, v_*, \omega, \tau) \tag{3.10}$$

Next, since $\Phi(v)=(1+|v|^2)^{-4}\ (\leqslant 1)$, Proposition 2 implies that

$$\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, G(v,v_*,\omega)\, d\omega\, dv_*\, dv<\infty, \qquad \forall t\in[0,\infty) \tag{3.11}$$

Thus by (3.10)

$$\lim_{n\to\infty} \tfrac{1}{4}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \times \Pi_f(v,v_*,\omega,\tau)\, \Gamma_n^-(F',F'_*,F,F_*)\, d\omega\, dv_*\, dv=0$$

and therefore by (3.8), (3.6)

$$\begin{aligned}\lim_{n\to\infty} \tfrac{1}{4}\int_0^t d\tau &\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \\ &\times \Pi_f(v,v_*,\omega,\tau)\, \Gamma_n^+(F',F'_*,F,F_*)\, d\omega\, dv_*\, dv \\ &=\lim_{n\to\infty}[S_{f_n}(t)-S_{f_n}(0)]=S_f(t)-S_f(0), \qquad \forall t\in[0,\infty)\end{aligned} \tag{3.12}$$

On the other hand, it is easily seen that for all $(v,v_*,\omega,\tau)\in\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2\times[0,\infty)$,

$$\begin{aligned}&\lim_{n\to\infty} \Gamma_n^+(F',F'_*,F,F_*)=\Gamma(F'F'_*,FF_*) \\ &\Pi_f(v,v_*,\omega,\tau)\,\Gamma(F'F'_*,FF_*) \\ &\qquad =\Gamma(f'f'_*(1+\varepsilon f)(1+\varepsilon f_*), ff_*(1+\varepsilon f')(1+\varepsilon f'_*))\end{aligned} \tag{3.13}$$

Since Γ and Γ_n^+ are nonnegative, by Fatou's lemma, (3.13) and (3.12) we obtain

$$\begin{aligned}\tfrac{1}{4}\int_0^t d\tau &\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \\ &\times \Gamma(f'f'_*(1+\varepsilon f)(1+\varepsilon f_*), ff_*(1+\varepsilon f')(1+\varepsilon f'_*))\, d\omega\, dv_*\, dv \\ &\leqslant \liminf_{n\to\infty} \tfrac{1}{4}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \\ &\times \Pi_f(v,v_*,\omega,\tau)\, \Gamma_n^+(F',F'_*,F,F_*)\, d\omega\, dv_*\, dv \\ &=S_f(t)-S_f(0)<\infty, \qquad \forall t\in[0,\infty)\end{aligned}$$

This integrability together with (3.13), (3.9), (3.11), (3.12) and dominated convergence theorem give the entropy identity (1.1):

$$\frac{1}{4}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)$$
$$\times \Gamma(f'f'_*(1+\varepsilon f)(1+\varepsilon f_*), ff_*(1+\varepsilon f')(1+\varepsilon f'_*))\, d\omega\, dv_*\, dv$$
$$= S_f(t) - S_f(0), \qquad \forall t\in[0,\infty)$$

The proof of the theorem is completed. ∎

4. EXISTENCE(II), MOMENT PRODUCTION, AND LOCAL STABILITY

We begin with a lemma which is an easy application of Fubini's theorem.

Lemma 7. Let $f_n = \bar{f}_n(|\cdot|)\in L^1(\mathbf{R}^3)$, $n=1,2,...$, i.e., f_n are isotropic functions. Then $\{f_n\}_{n=1}^{\infty}$ is weakly compact in $L^1(\mathbf{R}^3) \Leftrightarrow \{4\pi r^2 \bar{f}_n(r)\}_{n=1}^{\infty}$ is weakly compact in $L^1[0,\infty)$. Moreover $f_n \rightharpoonup f$ $(n\to\infty)$ weakly in $L^1(\mathbf{R}^3) \Leftrightarrow f=\bar{f}(|\cdot|)$ and $4\pi r^2\bar{f}_n(r) \rightharpoonup 4\pi r^2 \bar{f}(r)$ $(n\to\infty)$ weakly in $L^1[0,\infty)$.

Because of Lemma 7, we now introduce the family $\mathscr{R}$ of measurable radial sets of $\mathbf{R}^3$, i.e.,

$$E\in\mathscr{R} \Leftrightarrow \exists \text{ measurable set } \hat{E}\subset[0,\infty) \text{ such that } E=\{v\in\mathbf{R}^3 \mid |v|\in\hat{E}\}$$

and define, for any $\delta>0$,

$$\mathscr{R}_\delta = \{E\in\mathscr{R} \mid \operatorname{meas}(\hat{E})<\delta\}$$

Then for nonnegative isotropic functions $f=\bar{f}(|\cdot|)\in L^1(\mathbf{R}^3)$, define

$$V(f,\delta) = \sup_{E\in\mathscr{R}_\delta} \int_E f(v)\,dv$$
$$= \sup\left\{\int_{\hat{E}} 4\pi r^2 \bar{f}(r)\,dr \mid \hat{E}\subset[0,\infty),\ \operatorname{meas}(\hat{E})<\delta\right\} \tag{4.1}$$

Lemma 8. Suppose the kernel B satisfies (1.6)–(1.7). Let f be a nonnegative isotropic function in $L^1_2(\mathbf{R}^3)$. Then for any $\delta>0$,

$$\sup_{E\in\mathscr{R}_\delta}\int_E dv\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f'f'_*f_*\,d\omega\,dv_* \leqslant 8K\,\|f\|^2_{L^1}\,V(f,\delta) \qquad (4.2)$$

$$\sup_{E\in\mathscr{R}_\delta}\int_E dv\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f'f'_*\,d\omega\,dv_*$$
$$\leqslant [16\pi K\,\|f\|^2_{L^1}+4A^*\,\|f\|^2_{L^1_2}]\,\delta^{1/3} \qquad (4.3)$$

Proof. By Proposition 2 we have for any $E\in\mathscr{R}_\delta$

$$\int_E dv\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f(v')\, f(v'_*)\, f(v_*)\,d\omega\,dv_*$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f(v'_*)\, f(v')\, f(v)\, 1_{\hat{E}}(|v_*|)\,d\omega\,dv_*\,dv$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f(v'_*)\, f(v')\, f(v)$$
$$\times 1_{\hat{E}}(\sqrt{|v'|^2+|v'_*|^2-|v|^2})\,d\omega\,dv_*\,dv$$
$$\leqslant 4K\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} f(v)\, f(x)\, f(y)\, 1_{\hat{E}}(\sqrt{|x|^2+|y|^2-|v|^2})$$
$$\times 1_{\{|x|^2+|y|^2\geqslant|v|^2\}}\, 1_{\{|y|\geqslant|x|\}}\,dy\,dx\,dv$$
$$=4K\iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v)\, f(x)\left\{\int_{\mathbf{R}^3} f(y)\, 1_{\hat{E}_{v,x}}(|y|)\,dy\right\}dx\,dv \qquad (4.4)$$

where

$$\hat{E}_{v,x}=\{\rho\mid\rho\geqslant|x|,\ |x|^2+\rho^2\geqslant|v|^2 \text{ and } \sqrt{|x|^2+\rho^2-|v|^2}\in\hat{E}\}$$

Since

$$\operatorname{meas}(\hat{E}_{v,x})=\int_{|x|}^{\infty} 1_{\hat{E}}(\sqrt{|x|^2+\rho^2-|v|^2})\, 1_{\{|x|^2+\rho^2\geqslant|v|^2\}}\,d\rho$$
$$=\int_{\sqrt{(2|x|^2-|v|^2)\vee 0}}^{\infty} 1_{\hat{E}}(r)\,\frac{r}{\sqrt{r^2+|v|^2-|x|^2}}\,dr$$
$$\leqslant\int_{\sqrt{(2|x|^2-|v|^2)\vee 0}}^{\infty} 1_{\hat{E}}(r)\,\sqrt{2}\,dr\leqslant\sqrt{2}\,\delta$$

(where $a \vee b = \max\{a, b\}$), there exists an $R \geqslant 0$ such that $\text{meas}(\hat{E}_{v,x} \cap [0, R])) = \text{meas}(\hat{E}_{v,x} \cap (R, \infty)) = \frac{1}{2}\text{meas}(\hat{E}_{v,x}) < \delta$. Thus

$$\int_{\mathbf{R}^3} f(y)\, 1_{\hat{E}_{v,x}}(|y|)\, dy \leqslant 2V(f, \delta), \qquad v, x \in \mathbf{R}^3$$

Combining this with (4.4) leads to

$$\int_E dv \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f(v')\, f(v'_*)\, f(v_*)\, d\omega\, dv_* \leqslant 8K\, \|f\|^2_{L^1}\, V(f, \delta)$$

This proves (4.2). To prove (4.3), let $E \in \mathscr{R}_\delta$, $R = \delta^{-1/3}$. By equality $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ we see that $|v| > \sqrt{2}\, R$ implies $|v'| > R$ or $|v'_*| > R$. Thus if we define $f_R(v) = f(v)\, 1_{\{|v| > R\}}$ and $\phi^R(v) = 1_E(v)\, 1_{\{|v| \leqslant \sqrt{2}\, R\}}$, then

$$f(v')\, f(v'_*)\, 1_E(v) \leqslant f(v')\, f(v'_*)\, \phi^R(v) + f_R(v')\, f(v'_*) + f(v')\, f_R(v'_*)$$

Hence using Proposition 2, (3.3) and $1_E(v) = 1_{\hat{E}}(|v|)$, we have

$$\begin{aligned}
&\int_E dv \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f' f'_*\, d\omega\, dv_* \\
&\quad \leqslant \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f' f'_* \phi^R\, d\omega\, dv_*\, dv \\
&\qquad + \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)(f_R)'\, f'_*\, d\omega\, dv_*\, dv \\
&\qquad + \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f'(f_R)'_*\, d\omega\, dv_*\, dv \\
&\quad \leqslant 2K\, \|f\|^2_{L^1} \int_{\mathbf{R}^3} \phi^R(v)\, dv + 2 \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f_R f_*\, d\omega\, dv_*\, dv \\
&\quad \leqslant 2K\, \|f\|^2_{L^1}\, 4\pi \int_0^{\sqrt{2}\, R} r^2\, 1_{\hat{E}}(r)\, dr \\
&\qquad + 4A^* \iint_{\mathbf{R}^3 \times \mathbf{R}^3} f_R(v)\, f(v_*)(1 + |v|^2)^{1/2}\, (1 + |v_*|^2)^{1/2}\, dv_*\, dv \\
&\quad \leqslant 16\pi K\, \|f\|^2_{L^1}\, R^2 \delta + 4A^*\, \frac{1}{R} \int_{|v| > R} f(v)(1 + |v|^2)\, dv\, \|f\|_{L^1_1} \\
&\quad \leqslant 16\pi K\, \|f\|^2_{L^1}\, \delta^{1/3} + 4A^*\, \frac{1}{R}\, \|f\|^2_{L^1_2} = (16\pi K\, \|f\|^2_{L^1} + 4A^*\, \|f\|^2_{L^1_2})\, \delta^{1/3}
\end{aligned}$$

This proves (4.3). ∎

Lemma 9. Suppose the kernel B satisfies (1.6)–(1.7). Let $0 \leqslant f_0 \in L^1_2(\mathbf{R}^3)$ be an isotropic initial datum, and let f be a conservative isotropic solution of Eq. (BBE) with $f|_{t=0} = f_0$. Then for any $t \geqslant 0$ and any $\delta > 0$

$$V(f(\cdot, t), \delta) \leqslant [V(f_0, \delta) + C_1(f_0)\, \delta^{1/3} t] \exp(10\varepsilon K \|f_0\|^2_{L^1} t) \tag{4.5}$$

where $C_1(f_0) = 16\pi K \|f_0\|^2_{L^1} + 4A^* \|f_0\|^2_{L^1_2}$.

Proof. First of all, it is easily seen that the conditions (1.6)–(1.7) and Proposition 2 imply that

$$\|f(\cdot, t_1) - f(\cdot, t_2)\|_{L^1} \leqslant C(f_0)\, |t_1 - t_2|, \qquad t_1, t_2 \in [0, \infty) \tag{4.6}$$

where $C(f_0) := 4A^* \|f_0\|^2_{L^1_2} + 8\varepsilon K \|f_0\|^3_{L^1}$. This implies that for all measurable set $E \subset \mathbf{R}^3$,

$$\frac{d}{dt} \int_E f(v, t)\, dv = \int_{\mathbf{R}^3} Q(f, f)(v, t)\, 1_E(v)\, dv, \qquad t \in [0, \infty)$$

and the function $t \mapsto V(f(\cdot, t), \delta)$ is Lipschitz continuous:

$$|V(f(\cdot, t_1), \delta) - V(f(\cdot, t_2), \delta)| \leqslant C(f_0)\, |t_1 - t_2|, \qquad t_1, t_2 \in [0, \infty) \tag{4.7}$$

Now for any $E \in \mathscr{R}_\delta$, using Proposition 2 and Lemma 8 we have

$$\begin{aligned}
\frac{d}{dt} \int_E f(v, t)\, dv &\leqslant \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \\
&\quad \times f' f'_* (1 + \varepsilon f + \varepsilon f_*)\, 1_E(v)\, d\omega\, dv_*\, dv \\
&\leqslant (16\pi K \|f_0\|^2_{L^1} + 4A^* \|f_0\|^2_{L^1_2})\, \delta^{1/3} \\
&\quad + \varepsilon 2K \|f_0\|^2_{L^1} \int_E f(v, t)\, dv + \varepsilon 8K \|f_0\|^2_{L^1} V(f(\cdot, t), \delta) \\
&\leqslant C_1(f_0)\, \delta^{1/3} + 10\varepsilon K \|f_0\|^2_{L^1} V(f(\cdot, t), \delta)
\end{aligned}$$

After integration and taking $\sup_E$ we obtain for all $t \geqslant 0$

$$V(f(\cdot, t), \delta) \leqslant V(f_0, \delta) + C_1(f_0)\, \delta^{1/3} t + 10\varepsilon K \|f_0\|^2_{L^1} \int_0^t V(f(\cdot, \tau), \delta)\, d\tau$$

and the estimate (4.5) follows from Gronwall inequality. ∎

The following lemma is obvious (using spherical coordinate transformation).

Lemma 10. Let $0 \leqslant f = \bar{f}(|\cdot|) \in L^1(\mathbf{R}^3)$, and let $k(r) \geqslant 0$ be a nondecreasing function on $[0, \infty)$. Then

$$\int_{\mathbf{R}^3} f(v_*)\, k(|v - v_*|)\, dv_* \geqslant \frac{1}{2} \int_{\mathbf{R}^3} f(v_*)\, k(\sqrt{|v|^2 + |v_*|^2})\, dv_*, \qquad v \in \mathbf{R}^3$$

The following lemma gives a sharpened version of the Povzner's inequality, which is used in the moment estimates.

Lemma 11 (Lu[14]). Let $s > 2$, $0 \leqslant \gamma \leqslant \min\{s/2, 2\}$. Then for all $(v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$,

$$\begin{aligned} &|v'|^s + |v'_*|^s - |v|^s - |v_*|^s \\ &\quad \leqslant 2(2^{s/2} - 2)(|v|^{s-\gamma}\, |v_*|^\gamma + |v|^\gamma\, |v_*|^{s-\gamma}) - 2^{-s}(s/2 - 1)[\kappa(\theta)]^s\, |v|^s \end{aligned} \tag{4.8}$$

where $\kappa(\theta) = \min\{\cos\theta, 1 - \cos\theta\}$, $\theta = \arccos(|v - v_*|^{-1}\, |\langle v - v_*, \omega\rangle|)$. (For $v = v_*$, we define $\theta = 0$.)

For the angular function $b(\theta)$ in (1.8)–(1.9), the positive constants

$$A_s = 4\pi \int_0^{\pi/2} [\kappa(\theta)]^s \min\{(\cos\theta)^2 \sin\theta, b(\theta)\} \sin\theta\, d\theta, \qquad s \geqslant 0 \tag{4.9}$$

and A (in (1.9)) will be frequently used in the remainder of this paper.

In the following, for nonnegative measurable function g, we denote by $M_s(g)$ the $|v|$-moments of order s of g, i.e.,

$$M_s(g) = \int_{\mathbf{R}^3} g(v)\, |v|^s\, dv, \qquad s \geqslant 0$$

Recall that for conservative solutions f with $f|_{t=0} = f_0$ we have

$$M_0(f(\cdot, t)) = M_0(f_0) = M_0, \qquad M_2(f(\cdot, t)) = M_2(f_0) = M_2$$

Theorem 3. Suppose the kernel B satisfies (1.6)–(1.7). Let $0 \leqslant f_0 \in L_2^1(\mathbf{R}^3)$ be an isotropic initial datum with $M_0 > 0$. Then Eq. (BBE) has a conservative isotropic solution $f \in C^1([0, \infty), L^1(\mathbf{R}^3))$ with $f|_{t=0} = f_0$. Furthermore, if the kernal B satisfies (1.8)–(1.9), then the conservative isotropic solution f is also unique, and there exists a continuous increasing function $\Phi_f(r)$ on $[0, \infty)$ satisfying $\Phi_f(0) = 0$, such that for any conservative isotropic solution g (with $g_0 = g|_{t=0}$),

$$\|g(\cdot, t) - f(\cdot, t)\|_{L_2^1} \leqslant \Phi_f(\|g_0 - f_0\|_{L_2^1})\, e^{ct}, \qquad t \in [0, \infty) \tag{4.10}$$

where the constant $c > 0$ is independent of g.

Theorem 4. Suppose the kernel B satisfies (1.8)–(1.9). Let $0 \leqslant f_0 \in L_2^1(\mathbf{R}^3)$ be an isotropic initial datum with $M_0 > 0$, and let f be the unique conservative isotropic solution of Eq. (BBE) with $f|_{t=0} = f_0$. Then

(1) If $\beta > 0$, then for any $s > 2$

$$M_s(f(\cdot, t)) \leqslant M_2 \left(\frac{b_s}{1 - e^{-at}} \right)^{(s-2)/\beta}, \qquad t > 0 \tag{4.11}$$

where

$$a = 3\beta A M_0^{1-\beta/2} M_2^{\beta/2}, \qquad b_s = K^{-\beta/(3-\beta)} + \frac{2^{4s}}{A_s} [A(M_2/M_0)^{\beta/2} + \varepsilon K M_0] \tag{4.12}$$

(2) If $\beta = 0$ and $f_0 \in L_s^1(\mathbf{R}^3)$ for some $2 < s \leqslant 4$, then $f \in L^\infty([0, \infty); L_s^1(\mathbf{R}^3))$ and there exists $0 < t_s = t_s(f_0, A, \varepsilon, K, s, A_s) < \infty$ such that

$$M_s(f(\cdot, t)) \leqslant \left(2K^{-s/3} + \frac{2^{4s}}{A_s} (A + \varepsilon K M_0)(M_2/M_0)^{s/2} \right) M_0, \qquad t \geqslant t_s \tag{4.13}$$

Proof of Theorem 3 and Theorem 4

Part 1 (Existence and Moment Estimates). We first prove a weak stability property. Let $B_n(z, \omega)$ be collision kernels satisfying

$$0 \leqslant B_n(z, \omega) \leqslant B(z, \omega), \qquad \lim_{n \to \infty} B_n(z, \omega) = B(z, \omega), \qquad (z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2$$

and let $f_0^n \geqslant 0$ be isotropic initial data satisfying $f_0^n(v) \leqslant f_0(v)$ and $\lim_{n\to\infty} f_0^n(v) = f_0(v)$, $v \in \mathbf{R}^3$. Suppose f^n are conservative isotropic solutions of Eq. (BBE) corresponding to the kernels $B_n(z, \omega)$ with initial data $f^n|_{t=0} = f_0^n$. By Lemma 9 we have for any $t \geqslant 0$ and any $\delta > 0$,

$$\sup_{n\geqslant 1} V(f^n(\cdot, t), \delta) \leqslant [V(f_0, \delta) + C_1(f_0)\, \delta^{1/3} t] \exp(10\varepsilon K M_0^2 t) \tag{4.14}$$

Since f^n conserve the mass and energy, inequality (4.14) and Lemma 7 imply that for any $t \geqslant 0$, $\{f^n(\cdot, t)\}_{n=1}^{\infty}$ is weakly compact in $L^1(\mathbf{R}^3)$. By a diagonal process, there exists a subsequence $\{f^{n_k}\}_{k=1}^{\infty}$ such that for every rational number $t \in [0, \infty)$, $f^{n_k}(\cdot, t)$ converges weakly in $L^1(\mathbf{R}^3)$ to some $f(\cdot, t) \in L^1(\mathbf{R}^3)$ $(k \to \infty)$. Also by (4.6) we have

$$\sup_{n\geqslant 1} \|f^n(\cdot, t_1) - f^n(\cdot, t_2)\|_{L^1} \leqslant C(f_0)\, |t_1 - t_2|, \qquad t_1, t_2 \in [0, \infty)$$

Thus the weak convergence hold for all $t \in [0, \infty)$. For convenience we now suppose $f^n(\cdot, t)$ converges weakly in $L^1(\mathbf{R}^3)$ to $f(\cdot, t) \in L^1(\mathbf{R}^3)$ $(n \to \infty)$ for all $t \in [0, \infty)$. Obviously f is a nonnegative isotropic function and is measurable on $\mathbf{R}^3 \times [0, \infty)$ and satisfies $f|_{t=0} = f_0$, $f \in L^\infty([0, \infty); L^1_2(\mathbf{R}^3))$ and

$$\begin{gathered} \int_{\mathbf{R}^3} f(v, t)\, dv = \int_{\mathbf{R}^3} f_0(v)\, dv \\ \int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv \leqslant \int_{\mathbf{R}^3} f_0(v)\, |v|^2\, dv, \qquad t \in [0, \infty) \end{gathered} \tag{4.15}$$

Let $Q_n = Q_n^+ - Q_n^-$ be the collision operators corresponding to kernels B_n. Then for any isotropic function $\phi = \bar{\phi}(|\cdot|) \in L^\infty(\mathbf{R}^3)$ we have, by Proposition 2,

$$\begin{aligned} &\sup_{n\geqslant 1,\, t\geqslant 0} \int_{\mathbf{R}^3} Q_n^{\pm}(f^n, f^n)(v, t)\, |\phi(v)|\, dv \\ &\qquad \leqslant (2A^* \|f_0\|_{L^1_2}^2 + 4\varepsilon K \|f_0\|_{L^1_2}^3)\, \|\phi\|_{L^\infty(\mathbf{R}^3)} \end{aligned}$$

and by Proposition 3

$$\lim_{n\to\infty} \int_{\mathbf{R}^3} Q_n^{\pm}(f^n, f^n)(v, t)\, \phi(v)\, dv = \int_{\mathbf{R}^3} Q^{\pm}(f, f)(v, t)\, \phi(v)\, dv, \qquad t \geqslant 0$$

Therefore using dominated convergence theorem we have for any $t \geqslant 0$,

$$\int_{\mathbf{R}^3} f(v, t)\, \phi(v)\, dv = \lim_{n \to \infty} \int_{\mathbf{R}^3} f^n(v, t)\, \phi(v)\, dv$$

$$= \int_{\mathbf{R}^3} f_0(v)\, \phi(v)\, dv + \lim_{n \to \infty} \int_0^t d\tau \int_{\mathbf{R}^3} Q_n(f^n, f^n)(v, \tau)\, \phi(v)\, dv$$

$$= \int_{\mathbf{R}^3} f_0(v)\, \phi(v)\, dv + \int_0^t d\tau \int_{\mathbf{R}^3} Q(f, f)(v, \tau)\, \phi(v)\, dv$$

$$= \int_{\mathbf{R}^3} f_0(v)\, \phi(v)\, dv + \int_{\mathbf{R}^3} \left[\int_0^t Q(f, f)(v, \tau)\, d\tau \right] \phi(v)\, dv$$

and so $f(v, t) = f_0(v) + \int_0^t Q(f, f)(v, \tau)\, d\tau$ for $t \in [0, \infty)$, $v \in \mathbf{R}^3 \setminus Z_t$ with $\text{meas}(Z_t) = 0$. After a modification on v-null sets, f is a solution of Eq. (BBE) in the class $L^\infty([0, \infty); L^1_2) \cap C^1[0, \infty); L^1(\mathbf{R}^3))$. Also, (4.15) and Theorem 2 imply that f is a conservative solution.

Now we choose $B_n(z, \omega) = B(z, \omega) \wedge n$ and $f^n_0 \equiv f_0$. Then Theorem 1 and the weak stability imply the existence of conservative solution of Eq. (BBE). Moreover, if $f_0 \in L^1_s(\mathbf{R}^3)$ for some $s > 2$, then by Theorem 1, the approximate solutions f^n satisfy $\sup_{t \in [0, t_1], n \geqslant 1} \|f^n(\cdot, t)\|_{L^1_s} < \infty$, $\forall t_1 > 0$. Thus the conservative solution f, which is a weak limit of a subsequence of $\{f^n\}_{n=1}^{\infty}$, satisfies $\sup_{t \in [0, t_1]} \|f(\cdot, t)\|_{L^1_s} < \infty$, $\forall t_1 > 0$. If $f_0 \in L^1_2(\mathbf{R}^3)$, then applying this result to initial data $f_{n0}(v) = f_0(v)\, e^{-(1/n)\, |v|^2}$, we obtain conservative solutions f_n with $f_n\,|_{t=0} = f_{n0}$ satisfying $\sup_{t \in [0, t_1]} \|f_n(\cdot, t)\|_{L^1_s} < \infty$ for all $t_1 > 0$ and all $s > 2$, and therefore the function $t \mapsto M_s(f_n(\cdot, t))$ is in $C^1[0, \infty)$ for all $s > 2$, and

$$\frac{d}{dt} M_s(f_n(\cdot, t)) = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} Q(f_n, f_n)(v, t)\, |v|^s\, d\omega\, dv_*\, dv, \qquad t \in [0, \infty)$$

$n = 1, 2,\ldots$. Let f be the conservative solution which is a weak limit of a subsequence of $\{f_n\}_{n=1}^{\infty}$. By weak convergence we have

$$M_s(f(\cdot, t)) \leqslant \limsup_{n \to \infty} M_s(f_n(\cdot, t)), \qquad \forall t > 0, \quad \forall s > 2$$

Also we have $\lim_{n \to \infty}(M_0(f_{n0}), M_2(f_{n0})) = (M_0, M_2)$. Thus, to prove the moment estimate (4.11), we may assume that the initial datum f_0 and the solution f have the same properties as those of f_{n0} and f_n. Using the inequality (4.8) with $\gamma = \beta$ and $\gamma = 0$ respectively (the later is used for "Bose parts") we have for any $s > 2$,

$$\frac{d}{dt}M_s(f(\cdot,t))=\frac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)\,ff_*(|v'|^s+|v'_*|^s-|v|^s-|v|^s)$$
$$\times d\omega\,dv_*\,dv+\frac{1}{2}\varepsilon\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)$$
$$\times ff_*(f'+f'_*)(|v'|^s+|v'_*|^s-|v|^s-|v|^s)\,d\omega\,dv_*\,dv$$
$$\leqslant(2^{s/2}-2)\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)$$
$$\times ff_*(|v|^{s-\beta}\,|v_*|^{\beta}+|v|^{\beta}\,|v_*|^{s-\beta})\,d\omega\,dv_*\,dv$$
$$-2^{-s-1}(s/2-1)\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)$$
$$\times ff_*[\kappa(\theta)]^s\,|v|^s\,d\omega\,dv_*\,dv$$
$$+\varepsilon(2^{s/2}-2)\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)\,ff_*(f'+f'_*)$$
$$\times(|v|^s+|v_*|^s)\,d\omega\,dv_*\,dv$$
$$:=(2^{s/2}-2)\,I_1-2^{-s-1}(s/2-1)\,I_2+\varepsilon(2^{s/2}-2)\,I_3$$

Since $s>2\geqslant 2\beta$, we have $|v|^{s-\beta}\,|v_*|^{2\beta}\leqslant|v|^{\beta}\,|v_*|^s+|v|^s\,|v_*|^{\beta}$. This gives

$$I_1\leqslant 2A\iint_{\mathbf{R}^3\times\mathbf{R}^3}ff_*\,|v|^{s-\beta}\,|v_*|^{\beta}\,(|v|^{\beta}+|v_*|^{\beta})\,dv_*\,dv$$
$$\leqslant 6AM_{\beta}(f)(t)\,M_s(f(\cdot,t))\leqslant 6AM_2^{\beta/2}M_0^{1-\beta/2}M_s(f(\cdot,t))$$

For I_3, using Proposition 2 we have $I_3\leqslant 8KM_0^2M_s(f(\cdot,t))$. For I_2, using Lemma 10 and the equality $\min\{x,y\}=y-(y-x)^+$ we have

$$I_2\geqslant\tfrac{1}{2}A_s\iint_{\mathbf{R}^3\times\mathbf{R}^3}f\,|v|^s\,f_*(|v|^2+|v_*|^2)^{\beta/2}$$
$$\times\min\{K(|v|^2+|v_*|^2)^{(3-\beta)/2},1\}\,dv_*\,dv$$
$$\geqslant\tfrac{1}{2}A_sM_0\iint_{\mathbf{R}^3}f\,|v|^{s+\beta}\min\{K\,|v|^{3-\beta},1\}\,dv_*\,dv$$
$$=\tfrac{1}{2}A_sM_0\int_{\mathbf{R}^3}f\,|v|^{s+\beta}\,dv-\tfrac{1}{2}A_sM_0\int_{\mathbf{R}^3}f\,|v|^{s+\beta}\,(1-K\,|v|^{3-\beta})^+\,dv$$
$$\geqslant\tfrac{1}{2}A_sM_0\,M_{s+\beta}(f(\cdot,t))-\tfrac{1}{2}A_sM_0K^{-\beta/(3-\beta)}M_s(f(\cdot,t))$$

Further, by Hölder inequality, we have

$$[M_s(f(\cdot, t))]^{1+\beta/(s-2)} \leqslant M_{s+\beta}(f(\cdot, t))\, M_2^{\beta/(s-2)}$$

and so

$$I_2 \geqslant \tfrac{1}{2} A_s M_0 M_2^{-\beta/(s-2)} [M_s(f(\cdot, t))]^{1+\beta/(s-2)} - \tfrac{1}{2} A_s M_0 K^{-\beta/(3-\beta)} M_s(f(\cdot, t))$$

Therefore

$$\frac{d}{dt} M_s(f(\cdot, t)) \leqslant C_1 M_s(f(\cdot, t)) - C_2 [M_s(f(\cdot, t))]^{1+\beta/(s-2)}, \qquad t \geqslant 0$$

which implies

$$M_s(f(\cdot, t)) \leqslant \left[\frac{C_1}{C_2(1 - \exp\{-(\beta/(s-2))\, C_1 t\})} \right]^{(s-2)/\beta}, \qquad t > 0$$

where

$$\begin{aligned} C_1 &= (2^{s/2} - 2)\, 6 A M_2^{\beta/2} M_0^{1-\beta/2} + (2^{s/2} - 2)\, 8 \varepsilon K M_0^2 \\ &\quad + 2^{-s-2}(s/2 - 1)\, A_s M_0 K^{-\beta/(3-\beta)} \\ &> (s-2)\, 3 A M_2^{\beta/2} M_0^{1-\beta/2} = \frac{s-2}{\beta}\, a \end{aligned}$$

$$C_2 = 2^{-s-2}(s/2 - 1)\, A_s M_0 M_2^{-\beta/(s-2)}, \qquad \text{and} \qquad \frac{C_1}{C_2} \leqslant b_s M_2^{\beta/(s-2)}$$

This proves (4.11).

Now we prove the estimate (4.13) ($\beta = 0$, $2 < s \leqslant 4$). In Lemma 11, choose $\gamma = s/2 (\leqslant 2)$. Then using the same derivation above we have

$$\frac{d}{dt} M_s(f(\cdot, t)) \leqslant 2(2^{s/2} - 2)\, J_1 - 2^{-s-1}(s/2 - 1)\, J_2 + \varepsilon 2(2^{s/2} - 2)\, J_3$$

where

$$\begin{aligned} J_1 &= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f f_*\, |v|^{s/2}\, |v_*|^{s/2}\, d\omega\, dv_*\, dv \\ &\leqslant A[M_{s/2}(f(\cdot, t))]^2 \leqslant A M_2^{s/2} M_0^{2-s/2} \end{aligned}$$

$$\begin{aligned} J_3 &= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f f_* f'\, |v|^{s/2}\, |v_*|^{s/2}\, d\omega\, dv_*\, dv \\ &\leqslant 2 K M_0 [M_{s/2}(f)(t)]^2 \leqslant 2 K M_0 M_2^{s/2} M_0^{2-s/2} \end{aligned}$$

and (using Lemma 10 and $x \wedge y = y - (y-x)^+$ again)

$$J_2 = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v-v_*, \omega)\, ff_*\, |v|^s\, [\kappa(\theta)]^s\, d\omega\, dv_*\, dv$$

$$\geqslant \tfrac{1}{2} A_s \iint_{\mathbf{R}^3 \times \mathbf{R}^3} f(v,t)\, |v|^s\, f(v_*,t) \min\{K\, |v|^3, 1\}\, dv_*\, dv$$

$$= \tfrac{1}{2} A_s M_0 \int_{\mathbf{R}^3} f(v,t)\, |v|^s\, dv - \tfrac{1}{2} A_s M_0 \int_{\mathbf{R}^3} f(v,t)\, |v|^s\, (1 - K\, |v|^3)^+\, dv$$

$$\geqslant \tfrac{1}{2} A_s M_0 M_s(f(\cdot, t)) - \tfrac{1}{2} A_s K^{-s/3} M_0^2$$

Therefore

$$\frac{d}{dt} M_s(f(\cdot,t)) \leqslant C_3 - C_4 M_s(f(\cdot,t)), \ t \geqslant 0$$

This implies

$$M_s(f(\cdot,t)) \leqslant e^{-C_4 t} M_s(f_0) + \frac{C_3}{C_4}, \qquad t \geqslant 0$$

where

$$C_3 = 2(2^{s/2} - 2)\, A M_2^{s/2} M_0^{2-s/2} + 4(2^{s/2} - 2) \times \varepsilon K M_0 M_2^{s/2} M_0^{2-s/2} + 2^{-s-2}(s/2 - 1)\, A_s K^{-s/3} M_0^2$$

$$C_4 = 2^{-s-2}(s/2-1)\, A_s M_0$$

and

$$\frac{C_3}{C_4} \leqslant \frac{1}{2} \left(2K^{-s/3} + \frac{2^{4s}}{A_s} (A + \varepsilon K M_0)(M_2/M_0)^{s/2} \right) M_0 := C_5$$

Thus for

$$t_s = \max \left\{ 0, \frac{\log(M_s(f_0)/C_5)}{C_4} \right\} (< \infty)$$

we have $M_s(f(\cdot,t)) \leqslant 2C_5$, $t \geqslant t_s$. This proves (4.13).

Part 2 (Local Stability). In the case $\beta = 0$, the local stability (4.10) is obvious: from the proof of Theorem 1 and Gronwall inequality we have $\|g(\cdot,t) - f(\cdot,t)\|_{L^1_2} \leqslant \|g_0 - f_0\|_{L^1_2} e^{ct}$, $t \geqslant 0$, where the constant c can be so chosen that it depends only on $\|f_0\|_{L^1_2}$, A, and εK since f and g are both conservative solutions.

Now suppose $\beta > 0$. In this case, because of Proposition 2, the proof of (4.10) is essentially the same as that for the original Boltzmann equation (Lu[14]). To completeness, we present it as follows. It suffices to prove that the conservative isotropic solution f obtained in Part 1 (i.e., f satisfies the moment estimate (4.11)) satisfies (4.10) for all conservative isotropic solutions g. In fact, this implies the uniqueness of conservative isotropic solutions. Let

$$\Psi_f(r) = \sup_{0 \leqslant t \leqslant r} \int_{|v| > 1/\sqrt{r}} f(v, t)(1 + |v|^2)\, dv, \qquad r > 0; \qquad \Psi_f(0) = 0$$

Then using a generalized dominated convergence theorem [15, Theorem 3.4] it is easily shown that the non-decreasing function $\Psi_f(\cdot)$ is continuous on $[0, \infty)$ (see also ref. 14). Now we prove that the function $\Phi_f(r)$ in (4.10) can be taken as

$$\Phi_f(r) = C[r + \sqrt{r} + \Psi_f(r)], \qquad r \geqslant 0$$

where C is a positive constant depending only on f_0, β, ε, K, and the angular function $b(\cdot)$. In the following the same letters C, c will denote different such constants. Let

$$U_s(t) = \| g(\cdot, t) - f(\cdot, t) \|_{L^1_s}, \qquad t \geqslant 0, \quad 0 \leqslant s \leqslant 2$$

We prove that for any conservative isotropic solution g of Eq. (BBE),

$$U_2(t) \leqslant C[\, U_2(0) + \sqrt{U_2(0)} + \Psi_f(U_2(0))]\, e^{ct}, \qquad t \geqslant 0 \tag{4.16}$$

If $U_2(0) \geqslant 1$, then the conservation of the mass and energy implies that

$$U_2(t) \leqslant \| g_0 \|_{L^1_2} + \| f_0 \|_{L^1_2} \leqslant (1 + 2\, \| f_0 \|_{L^1_2})\, U_2(0), \qquad t \geqslant 0 \tag{4.17}$$

where $g_0 = g\,|_{t=0}$. Therefore, in the following, we assume that $U_2(0) < 1$ (which implies $\| g_0 \|_{L^1_2} \leqslant 1 + \| f_0 \|_{L^1_2}$ and $U_2(t) \leqslant 1 + 2\, \| f_0 \|_{L^1_2}$ for all $t \geqslant 0$). To prove (4.16), we need three inequalities: For any $0 < r \leqslant 1$,

$$U_2(t) \leqslant U_2(r) + C \int_r^t \left(1 + \frac{1}{\tau}\right) U_1(\tau)\, d\tau + C \int_r^t U_2(\tau)\, d\tau, \qquad t \geqslant r \tag{4.18}$$

$$U_2(t) \leqslant U_2(0) + \frac{4}{\sqrt{r}}\, U_1(t) + 2\Psi_f(r), \qquad 0 \leqslant t \leqslant r \tag{4.19}$$

$$U_1(t) \leqslant U_1(0) + C \int_0^t U_2(\tau)\, d\tau, \qquad t \geqslant 0 \tag{4.20}$$

(4.20) is obvious because of $0<\beta\leqslant 1$ and Proposition 2. Inequality (4.19) follows from the identity $|g-f| = g-f+2(f-g)^+$, the conservation of mass and energy, and the definition of $\Psi_f(\cdot)$. Also, for (4.18), we have

$$U_2(t) = \|g(\cdot, r)\|_{L^1_2} - \|f(\cdot, r)\|_{L^1_2} + 2\,\|[f(\cdot, t) - g(\cdot, t)]^+\|_{L^1_2}, \qquad t\geqslant r \tag{4.21}$$

Then applying the integral form of Eq. (BBE), we have for a.e. $v\in\mathbf{R}^3$,

$$\begin{aligned}[f(v,t)-g(v,t)]^+ = [f(v,r)-g(v,r)]^+ &+ \int_r^t d\tau \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\\ &\times\{[f'f'_* - g'g'_*)] - [ff_* - gg_*]\}\,1_{\{f>g\}}\,d\omega\,dv_*\\ &+\varepsilon\int_r^t d\tau \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\,1_{\{f>g\}}\\ &\times\{[f'f'_*(f+f_*) - g'g'_*(g+g_*)]\\ &-[ff_*(f'+f'_*) - gg_*(g'+g'_*)]\}\,d\omega\,dv_* \end{aligned} \tag{4.22}$$

Next, by the nonnegativity of f and g, it is easily shown that

$$\begin{aligned}&\{(f'f'_* - g'g'_*) - (ff_* - gg_*)\}\,1_{\{f>g\}}\\ &\quad\leqslant (f'f'_* - g'g'_*)^+ - (ff_* - gg_*)^+ + f\,|g_* - f_*| \end{aligned} \tag{4.23}$$

Since $(ff_* - gg_*)^+ \leqslant ff_*$, and the solution f satisfies the moment estimate (4.11) (choose $s=2+\beta$) which implies that for $t\geqslant r(>0)$

$$\begin{aligned}&\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\,ff_*(1+|v|^2)\,d\omega\,dv\,dv_*\\ &\quad\leqslant A\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} f(v,\tau)\,f(v_*,\tau)(1+|v|^2)\,|v-v_*|^\beta\,dv\,dv_*\\ &\quad\leqslant A\int_r^t \|f(\cdot,\tau)\|_{L^1_{2+\beta}}\,\|f(\cdot,\tau)\|_{L^1_\beta}\,d\tau\\ &\quad\leqslant A\,\|f_0\|_{L^1_2}\int_r^t 2[M_0 + M_2 M_{2+\beta}(f(\cdot,\tau))]\,d\tau < \infty\end{aligned}$$

it follows from $|v'|^2+|v'_*|^2=|v|^2+|v_*|^2$ that

$$\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(f'f'_*-g'g'_*)^+\,(1+|v|^2)\,d\omega\,dv\,dv_*$$

$$=\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(ff_*-gg_*)^+\,(1+|v|^2)\,d\omega\,dv\,dv_*<\infty$$

Therefore by (4.22), (4.23) and Proposition 2,

$$\begin{aligned}
&\|[f(\cdot,t)-g(\cdot,t)]^+\|_{L^1_2}\\
&\quad\leqslant \|[f(\cdot,r)-g(\cdot,r)]^+\|_{L^1_2}+\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\\
&\qquad\times f\,|g_*-f_*|\,(1+|v|^2)\,d\omega\,dv\,dv_*+C\int_r^t U_2(\tau)\,d\tau\\
&\quad\leqslant \|[f(\cdot,r)-g(\cdot,r)]^+\|_{L^1_2}\\
&\qquad+A\int_r^t \|f(\cdot,\tau)\|_{L^1_{2+\beta}}\,\|g(\cdot,\tau)-f(\cdot,\tau)\|_{L^1_\beta}\,d\tau+C\int_r^t U_2(\tau)\,d\tau\\
&\quad\leqslant \|[f(\cdot,r)-g(\cdot,r)]^+\|_{L^1_2}\\
&\qquad+C\int_r^t\left(1+\frac{1}{\tau}\right)U_1(\tau)\,d\tau+C\int_r^t U_2(\tau)\,d\tau,\qquad t\geqslant r
\end{aligned}$$

This estimate together with (4.21) gives (4.18).

In (4.18), choose $r=1$. Then, since $U_1(\cdot)\leqslant U_2(\cdot)$, it follows from Gronwall inequality that

$$U_2(t)\leqslant U_2(1)\,e^{c(t-1)},\qquad t\geqslant 1 \tag{4.24}$$

Now let $r>0$ satisfy $U_2(0)\leqslant r\leqslant 1$, and let $U^*(r)=\sup_{0\leqslant t\leqslant r}U_2(t)$. Then using (4.18), (4.20) and Fubini's theorem we have

$$\begin{aligned}
U_2(t)&\leqslant U_2(r)+C\int_r^t\frac{1}{\tau}\,U_1(\tau)\,d\tau+C\int_r^t U_2(\tau)\,d\tau\\
&\leqslant U_2(r)+CU_1(0)\,|\log r|+C\int_r^t\frac{1}{\tau}\int_0^\tau U_2(\sigma)\,d\sigma\,d\tau+C\int_r^t U_2(\tau)\,d\tau\\
&\leqslant U^*(r)+Cr\,|\log r|+C\int_0^t U_2(\sigma)(|\log\sigma|+1)\,d\sigma,\qquad t\in[r,1]
\end{aligned}\tag{4.25}$$

Since the last inequality in (4.25) also holds for $U_2(t)$ for $t \in [0, r]$, it follows from Gronwall inequality that

$$U_2(t) \leqslant C[U^*(r) + r\,|\log r|], \qquad t \in [0, 1] \tag{4.26}$$

For $U^*(r)$, we have, by (4.19) and (4.20),

$$\begin{aligned} U^*(r) &\leqslant U_2(0) + \frac{4}{\sqrt{r}}\left(U_1(0) + C\int_0^r U_2(\tau)\,d\tau\right) + 2\Psi_f(r) \\ &\leqslant C[r + \sqrt{r} + \Psi_f(r)] \end{aligned} \tag{4.27}$$

Therefore, combining (4.26), (4.27), (4.24) with (4.17) we obtain

$$U_2(t) \leqslant C[r + \sqrt{r} + \Psi_f(r)]\, e^{ct}, \qquad t \geqslant 0, \quad r > 0, \quad r \geqslant U_2(0)$$

This gives the estimate (4.16) by taking $r = U_2(0)$ for $U_2(0) > 0$ and by letting $r \to 0^+$ for $U_2(0) = 0$, respectively. The proofs of Theorem 3 and Theorem 4 are completed. ▮

5. LONG-TIME BEHAVIOR(I): LOW TEMPERATURE

This section deals with the velocity concentration at the very low temperature condition $T < T_b$. Some results of this section are also used in the next section. We first give the

Proof of Proposition 1. For any $0 < a \leqslant 1/\varepsilon$, $b > 0$, we have

$$\Omega_{a,b}(v) = \frac{1}{\varepsilon} \cdot \frac{\varepsilon a e^{-b|v|^2}}{1 - \varepsilon a e^{-b|v|^2}} = \frac{1}{\varepsilon} \sum_{n=1}^{\infty} (\varepsilon a)^n e^{-nb|v|^2}, \qquad v \in \mathbf{R}^3 \backslash \{0\}$$

and so

$$\begin{aligned} \int_{\mathbf{R}^3} \Omega_{a,b}(v)\,dv &= \frac{1}{\varepsilon} \sum_{n=1}^{\infty} (\varepsilon a)^n \int_{\mathbf{R}^3} e^{-nb|v|^2}\,dv \\ &= \frac{1}{\varepsilon}\left(\frac{1}{b}\right)^{3/2} \pi^{3/2} \sum_{n=1}^{\infty} (\varepsilon a)^n n^{-3/2} \end{aligned}$$

$$\begin{aligned} \int_{\mathbf{R}^3} \Omega_{a,b}(v)\,|v|^2\,dv &= \frac{1}{\varepsilon} \sum_{n=1}^{\infty} (\varepsilon a)^n \int_{\mathbf{R}^3} |v|^2 e^{-nb|v|^2}\,dv \\ &= \frac{1}{\varepsilon}\left(\frac{1}{b}\right)^{5/2} \frac{3}{2} \pi^{3/2} \sum_{n=1}^{\infty} (\varepsilon a)^n n^{-5/2} \end{aligned}$$

Suppose (a, b) is a solution of the moment equation system (1.3). Then

$$\frac{M_2}{(M_0)^{5/3}} \geqslant \varepsilon^{2/3} \frac{3\pi}{2} \frac{\sum_{n=1}^{\infty} (\varepsilon a)^n n^{-5/2}}{(\sum_{n=1}^{\infty} (\varepsilon a)^n n^{-3/2})^{5/3}} = \mathrm{R}(\varepsilon a) \tag{5.1}$$

where

$$\mathrm{R}(t) = \varepsilon^{2/3} \frac{3\pi}{2} \frac{\sum_{n=1}^{\infty} n^{-5/2} t^n}{(\sum_{n=1}^{\infty} n^{-3/2} t^n)^{5/3}}, \qquad 0 < t \leqslant 1$$

We need to prove that the function $\mathrm{R}(t)$ satisfies

$$\frac{d\mathrm{R}(t)}{dt} < 0, \qquad t \in (0, 1); \qquad \text{and} \qquad \lim_{t \to 0^+} \mathrm{R}(t) = \infty \tag{5.2}$$

The second property is obvious. Let $t \in (0, 1)$. By computing we see that

$$\frac{d\mathrm{R}(t)}{dt} < 0 \Leftrightarrow \left[\sum_{n=0}^{\infty} (n+1)^{-3/2} t^n \right]^2 < \frac{5}{3} \left[\sum_{n=0}^{\infty} (n+1)^{-5/2} t^n \right] \left[\sum_{n=0}^{\infty} (n+1)^{-1/2} t^n \right]$$

Since

$$\left[\sum_{n=0}^{\infty} (n+1)^{-3/2} t^n \right]^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (k+1)^{-3/2} (n+1-k)^{-3/2} t^n$$

and

$$\left[\sum_{n=0}^{\infty} (n+1)^{-5/2} t^n \right] \left[\sum_{n=0}^{\infty} (n+1)^{-1/2} t^n \right] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \tfrac{1}{2} [(k+1)^{-5/2} (n+1-k)^{-1/2} + (n+1-k)^{-5/2} (k+1)^{-1/2}] t^n$$

it suffices to to show that

$$(k+1)^{-3/2} (n+1-k)^{-3/2} < \tfrac{5}{6} [(k+1)^{-5/2} (n+1-k)^{-1/2} + (n+1-k)^{-5/2} (k+1)^{-1/2}]$$

$k = 0, 1, ..., n$; i.e.,

$$\left(\frac{1}{k+1}\right)\left(\frac{1}{n+1-k}\right) < \frac{5}{6}\left[\left(\frac{1}{k+1}\right)^2 + \left(\frac{1}{n+1-k}\right)^2\right], \qquad k = 0, 1, ..., n$$

But these inequalities hold obvoiusly. Therefore $d\mathrm{R}(t)/dt < 0$ for all $t \in (0, 1)$. Since $\varepsilon a \in (0, 1]$, it follows from (5.1) that M_0, M_2 satisfy the condition (1.4), i.e.,

$$\frac{M_2}{(M_0)^{5/3}} \geqslant \mathrm{R}(1) = \varepsilon^{2/3}\,\frac{3}{2\pi}\,\frac{\zeta(5/2)}{[\zeta(3/2)]^{5/3}}$$

Conversely, if M_0, M_2 satisfy (1.4), then by (5.2) we see that there exists a unique $0 < a \leqslant 1/\varepsilon$ such that $M_2/(M_0)^{5/3} = \mathrm{R}(a\varepsilon)$. Therefore with

$$b = \left\{(\varepsilon M_0)^{-1}\,\pi^{3/2} \sum_{n=1}^{\infty} (\varepsilon a)^n\, n^{-3/2}\right\}^{2/3} \quad (>0)$$

(a, b) is a unique solution of the moment equation system (1.5). ∎

In order to show the velocity concentration, we need the following Lemma 12 which is a consequence of Chacon's biting lemma (Brooks and Chacon[3]). One version we used of the Chacon's biting lemma is the following form of Ball and Murat[2] (see also Zhang[19]):

Let $(\Omega, \mathscr{F}, \mu)$ be a finite positive measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $L^1(\Omega, d\mu)$ i.e., $\sup_{n\geqslant 1} \int_\Omega |f_n|\, d\mu < \infty$. Then there exist a function $f \in L^1(\Omega, d\mu)$ and a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that

$$f_{n_j} \rightharpoonup f \ (j \to \infty) \text{ biting-weakly in } L^1(\Omega, d\mu)$$

That is, there exists a non-increasing sequence of sets $E_k \in \mathscr{F}$, with $\lim_{k\to\infty} \mu(E_k) = 0$, such that $f_{n_j} \rightharpoonup f$ $(j \to \infty)$ weakly in $L^1(\Omega \backslash E_k, d\mu)$ for every fixed k. ∎

Lemma 12. Let $\Omega \subset \mathbf{R}^N$ be a measurable set with $\mathrm{meas}(\Omega) = \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $L^1(\Omega)$ satisfying

$$\sup_{n\geqslant 1} \int_{\Omega\backslash B_R} |f_n(x)|\, dx \to 0 \qquad \text{as} \quad R \to \infty$$

where $B_R = \{x \in \mathbf{R}^N \mid |x| < R\}$. Then there exist a function $f_\infty \in L^1(\Omega)$ and a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that

$$f_{n_j} \rightharpoonup f_\infty \ (j \to \infty) \text{ biting-weakly in } L^1(\Omega)$$

Proof. Consider a positive measure μ defined on the Lebesgue measurable sets:

$$d\mu = (1+|x|^2)^{-N}\, dx, \qquad \text{i.e.,} \quad \mu(E) = \int_E (1+|x|^2)^{-N}\, dx$$

It is easily seen that μ is finite (i.e., $\mu(\Omega) < \infty$) and for any $Z \subset \Omega$, $\mu(Z) = 0 \Leftrightarrow \text{meas}(Z) = 0$ which implies that $L^\infty(\Omega \backslash E, d\mu) = L^\infty(\Omega \backslash E)$ for all measurable sets $E \subset \Omega$. Let $g_n(x) = f_n(x)(1+|x|^2)^N$. Then $\|g_n\|_{L^1(\Omega, d\mu)} = \|f_n\|_{L^1(\Omega)}$. By Chacon's biting lemma, there exist a function $g \in L^1(\Omega, d\mu)$, a subsequence $\{g_{n_j}\}_{j=1}^\infty$ and a non-increasing sequence of sets $\tilde{E}_l$, with $\lim_{l\to\infty} \mu(\tilde{E}_l) = 0$, such that $g_{n_j} \rightharpoonup g$ weakly in $L^1(\Omega \backslash \tilde{E}_l, d\mu)$ as $j \to \infty$ for every fixed l. Since $L^\infty(\Omega \backslash \tilde{E}_l, d\mu) = L^\infty(\Omega \backslash \tilde{E}_l)$, this is equivalent to $f_{n_j} \rightharpoonup f_\infty$ weakly in $L^1(\Omega \backslash \tilde{E}_l)$ as $j \to \infty$ for every fixed l, where $f_\infty(x) = g(x)(1+|x|^2)^{-N}$. Choose a subsequence $\{\tilde{E}_{l_i}\}_{i=1}^\infty$ such that $\mu(\tilde{E}_{l_i}) \leqslant i^{-4N}$, $i = 1, 2, \ldots$, and let $E_k = \bigcup_{i=k}^\infty (\tilde{E}_{l_i} \cap B_i)$. Then $E_1 \supset E_2 \supset E_3 \supset \cdots$, and $\text{meas}(\tilde{E}_{l_i} \cap B_i) \leqslant (1+i^2)^N \mu(\tilde{E}_{l_i} \cap B_i) \leqslant 2^N i^{-2N}$ so that $\text{meas}(E_k) \leqslant 2^N \sum_{i=k}^\infty i^{-2N} \to 0$ $(k \to \infty)$. Now given any $k \geqslant 1$ and any $\phi \in L^\infty(\Omega \backslash E_k)$. By defining $\phi = 0$ on E_k, $\phi \in L^\infty(\Omega)$. For any integer $R > k$, we have $E_k \supset (\tilde{E}_{l_R} \cap B_R)$, and so there is a set $\Delta_{k,R}$ such that $\Omega \backslash E_k = [(\Omega \backslash \tilde{E}_{l_R}) \cup (\tilde{E}_{l_R} \backslash B_R)] \cap \Delta_{k,R}$. Let $\phi_{k,R}(x) = \phi(x)\, 1_{\Delta_{k,R}}(x)$. Then

$$\begin{aligned}
&\left| \int_{\Omega \backslash E_k} f_{n_j}(x)\, \phi(x)\, dx - \int_{\Omega \backslash E_k} f_\infty(x)\, \phi(x)\, dx \right| \\
&\quad \leqslant \left| \int_{\Omega \backslash \tilde{E}_{l_R}} f_{n_j}(x)\, \phi_{k,R}(x)\, dx - \int_{\Omega \backslash \tilde{E}_{l_R}} f_\infty(x)\, \phi_{k,R}(x)\, dx \right| \\
&\qquad + \|\phi\|_{L^\infty(\Omega)} \left(\sup_{n \geqslant 1} \int_{\Omega \backslash B_R} |f_n(x)|\, dx + \int_{\Omega \backslash B_R} |f_\infty(x)|\, dx \right)
\end{aligned}$$

Thus first letting $j \to \infty$ and then letting $R \to \infty$ we obtain $\lim_{j\to\infty} \int_{\Omega \backslash E_k} f_{n_j} \phi\, dx = \int_{\Omega \backslash E_k} f_\infty \phi\, dx$. This proves the lemma. ∎

Theorem 5. Suppose the kernel $B(z, \omega)$ satisfies (1.6)–(1.7) and $B(z, \omega) > 0$ for all $0 < |\langle z, \omega \rangle| < |z|$. Let $0 \leqslant f_0 \in L_2^1(\mathbf{R}^3)$ be an isotropic initial datum satisfying $M_0 > 0$, and let f be a conservative isotropic solution of Eq. (BBE) with $f|_{t=0} = f_0$. Then for any sequence $\{t_n\}_{n=1}^\infty \subset [0, \infty)$ satisfying $\lim_{n\to\infty} t_n = \infty$, there exist a subsequence $\{t_{n_j}\}_{j=1}^\infty$ and an equilibrium solution $\Omega_{a,b}$ $(0 \leqslant a \leqslant 1/\varepsilon,\ b > 0)$ such that

$$4\pi r^2 \bar{f}(r, t_{n_j}) \rightharpoonup 4\pi r^2 \bar{\Omega}_{a,b}(r)\ (j \to \infty) \text{ biting-weakly in } L^1[0, \infty)$$

(where $\bar{\Omega}_{a,b}(|\cdot|)=\Omega_{a,b}$) and $\Omega_{a,b}$ must satisfy

$$\frac{1}{M_0}\int_{\mathbf{R}^3}\Omega_{a,b}(v)\,dv\leqslant \min\left\{1,\left(\frac{T}{T_b}\right)^{3/5}\right\} \tag{5.3}$$

As a consequence, if $\{f(\cdot,t_{n_j})\}_{j=1}^{\infty}$ is weakly convergent in $L^1(\mathbf{R}^3)$, then its weak limit must be an equilibrium solution $\Omega_{a,b}$ with $0<a\leqslant 1/\varepsilon$, $b>0$, and therefore M_0 and M_2 must satisfy the temperature condition $T\geqslant T_b$.

Proof. By conservation of mass and energy, we have $\int_0^\infty 4\pi r^2\bar{f}(r,t_n)(1+r^2)\,dr=M_0+M_2(<\infty)$. Thus by Lemma 12, there exist a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$ and a function $4\pi r^2\bar{f}_\infty(r)\in L^1[0,\infty)$ such that $4\pi r^2\bar{f}(r,t_{n_j})\rightharpoonup 4\pi r^2\bar{f}_\infty(r)$ $(j\to\infty)$ biting-weakly in $L^1[0,\infty)$. We now prove that the function $f_\infty(v):=\bar{f}_\infty(|v|)$ is an equilibrium solution of Eq. (BBE). For notation convenience, we may suppose that $4\pi r^2\bar{f}(r,t_n)\rightharpoonup 4\pi r^2\bar{f}_\infty(r)$ $(n\to\infty)$ biting-weakly in $L^1[0,\infty)$, i.e., there exist a non-increasing sequence of measurable sets $\hat{E}_k\subset[0,\infty)$ with $\lim_{k\to\infty}\text{meas}(\hat{E}_k)=0$, such that $4\pi r^2\bar{f}(r,t_n)\rightharpoonup 4\pi r^2\bar{f}_\infty(r)$ $(j\to\infty)$ weakly in $L^1([0,\infty)\backslash\hat{E}_k)$ for every fixed k. This implies first that that

$$\int_{\mathbf{R}^3} f_\infty(v)\,dv\leqslant M_0,\qquad \int_{\mathbf{R}^3} f_\infty(v)\,|v|^2\,dv\leqslant M_2 \tag{5.4}$$

And we may assume that $0\leqslant f_\infty(v)<\infty$ for all $v\in\mathbf{R}^3$. Then we consider

$$D(g)=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\times\Upsilon(g'g'_*(1+\varepsilon g+\varepsilon g_*)-gg_*(1+\varepsilon g'+\varepsilon g'_*))\,d\omega\,dv_*\,dv$$

where

$$\Upsilon(y)=\frac{y^2}{1+|y|},\qquad y\in\mathbf{R}$$

To prove that f_∞ is an equilibrium solution, it suffices to prove that $D(f_\infty)=0$. In fact, if we define $F=f_\infty/(1+\varepsilon f_\infty)$, then, since the kernel $B(v-v_*,\omega)>0$ a.e. on $\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2$, $D(f_\infty)=0$ implies that $F'F'_*=FF_*$ a.e. on $\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2$. Thus by a well-known result of Arkeryd [1, p. 26], if $\|f_\infty\|_{L^1}>0$ then there exist constants $a>0$, $b>0$ such that $F(v)=ae^{-b|v|^2}$ a.e. $v\in\mathbf{R}^3$, and therefore $f_\infty(v)=\Omega_{a,b}(v)$ a.e. $v\in\mathbf{R}^3$. Also the nonnegativity of f_∞ implies that $a\leqslant 1/\varepsilon$. If $\|f_\infty\|_{L^1}=0$, we choose $a=0$.

Now we prove $D(f_\infty)=0$. Since $f\in L^\infty([0,\infty),L^1_2(\mathbf{R}^3))$, it follows from Lemma 6 and Theorem 2 that the entropy $t\mapsto S_f(t)$ is continuous, bounded and monotonically non-decreasing on $[0, \infty)$. Thus for any $n \geqslant 1$, there exists $\delta_n > 0$ such that

$$\delta_n = \frac{1}{n} + \sqrt{S_f(t_n + \delta_n) - S_f(t_n)}$$

and $\lim_{n\to\infty} \delta_n = 0$ since $\lim_{n\to\infty} t_n = \infty$. Then from the entropy identity (1.1), for any $n \geqslant 1$, there exists $\tilde{t}_n \in [t_n, t_n + \delta_n]$ such that with $f_n(v) := f(v, \tilde{t}_n)$,

$$0 \leqslant e(f_n) \leqslant 4[S_f(t_n + \delta_n) - S_f(t_n)]/\delta_n \leqslant 4\delta_n$$

where

$$e(g) = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times \Gamma(g'g'_*(1 + \varepsilon g)(1 + \varepsilon g_*), gg_*(1 + \varepsilon g')(1 + \varepsilon g'_*))\, d\omega\, dv_*\, dv$$

This gives $\lim_{n\to\infty} e(f_n) = 0$. On the other hand, by (4.6) we have

$$\int_{\mathbf{R}^3} |f_n(v) - f(v, t_n)|\, dv \leqslant C(f_0)\, |\tilde{t}_n - t_n| \leqslant C(f_0)\, \delta_n \to 0 \qquad (n \to \infty)$$

This implies that $4\pi r^2 \bar{f}_n(r) \rightharpoonup 4\pi r^2 \bar{f}_\infty(r)$ biting-weakly in $L^1[0, \infty)$ as $n \to \infty$. Next let

$$\Phi_k(|v|, |v_*|, |v'|, |v'_*|) = 1_{[0,\infty)\setminus\hat{E}_k}(|v|)\, 1_{[0,\infty)\setminus\hat{E}_k}(|v_*|) \times 1_{[0,\infty)\setminus\hat{E}_k}(|v'|)\, 1_{[0,\infty)\setminus\hat{E}_k}(|v'_*|)$$

$$e_k(g) = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times \Gamma(g'g'_*(1 + \varepsilon g)(1 + \varepsilon g_*), gg_*(1 + \varepsilon g')(1 + \varepsilon g'_*)) \times \Phi_k\, d\omega\, dv_*\, dv$$

$$D_k(g) = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times \Upsilon(g'g'_*(1 + \varepsilon g + \varepsilon g_*) - gg_*(1 + \varepsilon g' + \varepsilon g'_*)) \times \Phi_k\, d\omega\, dv_*\, dv$$

and

$$f_n^k(v) = f_n(v)\, 1_{[0,\infty)\setminus\hat{E}_k}(|v|), \qquad f_\infty^k(v) = f_\infty(v)\, 1_{[0,\infty)\setminus\hat{E}_k}(|v|)$$

Then we have

$$e_k(f_n^k) = e_k(f_n) \leqslant e(f_n), \qquad D_k(f_\infty^k) = D_k(f_\infty)$$

Here we have used the convention that $\infty \cdot 0 = 0$. Let $\hat{Z} = \bigcap_{k=1}^{\infty} \hat{E}_k$,

$$\begin{aligned}&\Phi_\infty(|v|, |v_*|, |v'|, |v'_*|)\\ &\quad = 1_{[0,\infty)\setminus\hat{Z}}(|v|)\, 1_{[0,\infty)\setminus\hat{Z}}(|v_*|)\, 1_{[0,\infty)\setminus\hat{Z}}(|v'|)\, 1_{[0,\infty)\setminus\hat{Z}}(|v'_*|)\end{aligned}$$

Since $\hat{E}_1 \supset \hat{E}_2 \supset \hat{E}_3 \supset \cdots$, it follows that Φ_k converge non-decreasingly to Φ_∞ on $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$ and so by monotone convergence theorem, Fubini's theorem and meas$(\hat{Z}) = 0$ we have

$$D(f_\infty) = \lim_{k\to\infty} D_k(f_\infty) = \lim_{k\to\infty} D_k(f_\infty^k)$$

Thus we need only to prove that $D_k(f_\infty^k) = 0$ for all k. It should be noted that the biting-weak convergence and Lemma 7 implies that

$$f_n^k \rightharpoonup f_\infty^k \ (n \to \infty) \text{ weakly in } L^1(\mathbf{R}^3)$$

for every fixed k. Next observe that Υ is convex on $\mathbf{R}$ $(d^2\Upsilon(y)/dy^2 = 2(1+|y|^2)^{-3})$ and

$$\frac{d\Upsilon(y)}{dy} = \frac{y(2+|y|)}{(1+|y|)^2} := \Upsilon_1(y), \qquad |\Upsilon_1(y)| \leqslant 1, \quad y \in \mathbf{R}$$

Thus for bounded functions

$$\begin{aligned}\phi_k(|v|, |v_*|, |v'|, |v'_*|) = \Upsilon_1(&(f_\infty^k)'\,(f_\infty^k)'_*\,(1 + \varepsilon f_\infty^k + \varepsilon(f_\infty^k)_*)\\ &- f_\infty^k (f_\infty^k)_*\,(1 + \varepsilon(f_\infty^k)' + \varepsilon(f_\infty^k)'_*))\end{aligned}$$

we have, by convexity of Υ,

$$\begin{aligned}&\Upsilon((f_n^k)'\,(f_n^k)'_*\,(1 + \varepsilon f_n^k + \varepsilon(f_n^k)_*) - f_n^k (f_n^k)_*\,(1 + \varepsilon(f_n^k)' + \varepsilon(f_n^k)'_*))\\ &\quad \geqslant \Upsilon((f_\infty^k)'\,(f_\infty^k)'_*\,(1 + \varepsilon f_\infty^k + \varepsilon(f_\infty^k)_*)\\ &\qquad - f_\infty^k (f_\infty^k)_*\,(1 + \varepsilon(f_\infty^k)' + \varepsilon(f_\infty^k)'_*))\\ &\qquad + \phi_k(|v|, |v_*|, |v'|, |v'_*|)\{(f_n^k)'\,(f_n^k)'_*\,(1 + \varepsilon f_n^k + \varepsilon(f_n^k)_*)\\ &\qquad - f_n^k (f_n^k)_*\,(1 + \varepsilon(f_n^k)' + \varepsilon(f_n^k)'_*)\\ &\qquad - [(f_\infty^k)'\,(f_\infty^k)'_*\,(1 + \varepsilon f_\infty^k + \varepsilon(f_\infty^k)_*)\\ &\qquad - f_\infty^k (f_\infty^k)_*\,(1 + \varepsilon(f_\infty^k)' + \varepsilon(f_\infty^k)'_*)]\}\end{aligned}$$

and so by $\phi_k(|v'|, |v'_*|, |v|, |v_*|) = -\phi_k(|v|, |v_*|, |v'|, |v'_*|)$ we have

$$D_k(f^k_n) \geqslant D_k(f^k_\infty) + 2 \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)$$
$$\times \{(f^k_n)'\,(f^k_n)'_*\,(1 + \varepsilon f^k_n + \varepsilon (f^k_n)_*)$$
$$-(f^k_\infty)'\,(f^k_\infty)'_*\,(1 + \varepsilon f^k_\infty + \varepsilon (f^k_\infty)_*)\}\;\phi_k \Phi_k\, d\omega\, dv_*\, dv$$

By Proposition 3, the integral tends to zero as $n \to \infty$. Thus we obtain

$$D_k(f^k_\infty) \leqslant \liminf_{n \to \infty} D_k(f^k_n) \tag{5.5}$$

Next, in the following elementary inequality

$$(a - b)^2 \leqslant (a \vee b)\, \Gamma(a, b), \qquad a, b \in [0, \infty)$$

choose

$$a = (f^k_n)'\,(f^k_n)'_*\,(1 + \varepsilon f^k_n)(1 + \varepsilon (f^k_n)_*)$$
$$b = f^k_n (f^k_n)_*\,(1 + \varepsilon (f^k_n)')(1 + \varepsilon (f^k_n)'_*)$$

and let

$$\max = \max\{(f^k_n)', (f^k_n)'_*, f^k_n, (f^k_n)_*\}$$

Then for any $R > 1$,

$$\Upsilon(a - b) \leqslant R^2 (1 + \varepsilon R)^2\, \Gamma(a, b) + |(f^k_n)'\,(f^k_n)'_*\,(1 + \varepsilon f^k_n + \varepsilon (f^k_n)_*)$$
$$- f^k_n (f^k_n)_*\,(1 + \varepsilon (f^k_n)' + \varepsilon (f^k_n)'_*)|\; 1_{\{\max > R\}}$$

where we have used the reduction identity (1.10). By definition of $D_k(\cdot)$, this gives

$$D_k(f^k_n) \leqslant R^2 (1 + \varepsilon R)^2\, e_k(f^k_n) + 2 \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)(f^k_n)'\,(f^k_n)'_*$$
$$\times (1 + \varepsilon f^k_n + \varepsilon (f^k_n)_*)\; 1_{\{\max > R\}}\, d\omega\, dv_*\, dv \tag{5.6}$$

Further, by $1 < R$ we have, for instance,

$$(f^k_n)'\,(f^k_n)'_*\,(1 + \varepsilon f^k_n + \varepsilon (f^k_n)_*)\; 1_{\{\max = f^k_n > R\}}$$
$$\leqslant (1 + 2\varepsilon)(f^k_n)'\,(f^k_n)'_*\, f^k_n\, 1_{\{f^k_n > R\}}$$

Thus, by Proposition 2 and conservation of mass and energy, the second term in the right hand side of (5.6)

$$\begin{aligned}
&\leqslant 4 \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(f_n^k)'\,(f_n^k)'_*\\
&\quad\times(1+\varepsilon f_n^k+\varepsilon(f_n^k)_*)\,1_{\{\max=(f_n^k)'>R\}}\,d\omega\,dv_*\,dv\\
&\quad+4\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(f_n^k)'\,(f_n^k)'_*\\
&\quad\times(1+\varepsilon f_n^k+\varepsilon(f_n^k)_*)\,1_{\{\max=f_n^k>R\}}\,d\omega\,dv_*\,dv\\
&\leqslant 8A^*\,\|f_0\|_{L^1_2}\,\|f_n^k 1_{\{f_n^k>R\}}\|_{L^1_1}\\
&\quad+(8+32\varepsilon)\,KM_0^2\,\|f_n^k 1_{\{f_n^k>R\}}\|_{L^1}:=\Lambda_{k,n}(R)\leqslant\Lambda_k(R)
\end{aligned}$$

where $\Lambda_k(R)=\sup_{n\geqslant 1}\Lambda_{k,n}(R)$. Therefore

$$D_k(f_n^k)\leqslant R^2(1+\varepsilon R)^2\,e(f_n)+\Lambda_k(R)$$

Since it is easily seen that $f_n^k(v)(1+|v|^2)^{1/2}$ converge weakly to $f_\infty^k(v)(1+|v|^2)^{1/2}$ in $L^1(\mathbf{R}^3)$, the set $\{f_n^k(v)(1+|v|^2)^{1/2}\}_{n=1}^{\infty}$ is weakly compact in $L^1(\mathbf{R}^3)$. Thus $\Lambda_k(R)\to 0$ as $R\to\infty$. Therefore, first letting $n\to\infty$, then letting $R\to\infty$, we obtain $\lim_{n\to\infty} D_k(f_n^k)=0$. This implies $D_k(f_\infty^k)=0$ for all k by (5.5). Therefore, $f_\infty=\Omega_{a,b}$ $(0\leqslant a\leqslant 1/\varepsilon,\ b>0)$.

Now let $\lambda=(1/M_0)\int_{\mathbf{R}^3}\Omega_{a,b}(v)\,dv$. Then (5.4) shows that $\lambda\leqslant 1$ and

$$\int_{\mathbf{R}^3}\Omega_{a,b}(v)\,dv=\lambda M_0,\qquad \int_{\mathbf{R}^3}\Omega_{a,b}(v)\,|v|^2\,dv\leqslant M_2$$

If $\lambda>0$, then by Proposition 1 we have

$$\frac{M_2}{(\lambda M_0)^{5/3}}\geqslant\frac{3}{2\pi}\frac{\zeta(5/2)}{[\zeta(3/2)]^{5/3}}\,\varepsilon^{2/3}$$

Since

$$\frac{T}{T_b}=\frac{M_2}{(M_0)^{5/3}}\,\frac{2\pi[\zeta(3/2)]^{5/3}}{3\quad\zeta(5/2)}\,\frac{1}{\varepsilon^{2/3}}\tag{5.7}$$

this gives $\lambda^{5/3}\leqslant T/T_b$ and (5.3) holds. The remainder of the theorem is obvious. ∎

Theorem 6. Suppose the kernel $B(z,\omega)$ satisfies (1.6)–(1.7) and $B(z,\omega)>0$ for all $0<|\langle z,\omega\rangle|<|z|$. Let $f_0\geqslant 0$ be an isotropic initial datum in $L^1_2(\mathbf{R}^3)$ satisfying $M_0>0$ such that M_0, M_2 satisfy the very low temperature condition: $T<T_b$. Let f be a conservative isotropic solution of Eq. (BBE) with $f|_{t=0}=f_0$. Then for any $0<\epsilon<1/2$, there exist an $R>0$, a $t_0>0$ and a family $\{S_t\}_{t\geqslant t_0}$ of measurable radial sets $S_t\subset B_R$ with

$$\lim_{t\to\infty}\operatorname{meas}(S_t)=0$$

such that

$$\frac{1}{M_0}\int_{S_t} f(v,t)\,dv>(1-\epsilon)\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right],\qquad \forall t\geqslant t_0$$

Proof. Given $0<\epsilon<1/2$. We first prove that $\forall\delta>0$, $\exists\tau(\delta)>0$ such that (recall (4.1))

$$V(f(\cdot,t),\delta)\geqslant(1-\epsilon/3)\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right]M_0,\qquad \forall t\geqslant\tau(\delta)\tag{5.8}$$

Suppose, to the contrary, that there exists a $\delta_0>0$ such that $\forall\tau>0$, $\exists t_*\geqslant\tau$ such that $V(f(\cdot,t_*),\delta_0)<(1-\epsilon/3)[1-(T/T_b)^{3/5}]$. Then there is a sequence $\{t_n\}_{n=1}^{\infty}\subset[0,\infty)$ satisfying $\lim_{n\to\infty}t_n=\infty$ such that

$$V(f(\cdot,t_n),\delta_0)<(1-\epsilon/3)\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right]M_0,\qquad n=1,2,3,\dots\tag{5.9}$$

On the other hand, by Theorem 5, there exist a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$, an equilibrium solution $\Omega_{a,b}$ and a non-increasing sequence of measurable sets $\hat{E}_k\subset[0,\infty)$, with $\lim_{k\to\infty}\operatorname{meas}(\hat{E}_k)=0$, such that $4\pi r^2\bar{f}(r,t_{n_j})\rightharpoonup 4\pi r^2\bar{\Omega}_{a,b}(r)$ $(j\to\infty)$ weakly in $L^1([0,\infty)\backslash\hat{E}_k)$ for every fixed k, and $\Omega_{a,b}$ satisfies the inequality (5.3). Since f conserves the mass, it follows from (5.3) that

$$\begin{aligned}\int_{\hat{E}_k}4\pi r^2\bar{f}(r,t_{n_j})\,dr&=M_0-\int_{[0,\infty)\backslash\hat{E}_k}4\pi r^2\bar{f}(r,t_{n_j})\,dr\\&\to M_0-\int_{[0,\infty)\backslash\hat{E}_k}4\pi r^2\bar{\Omega}_{a,b}(r)\,dr\\&\geqslant\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right]M_0\qquad\text{as}\quad j\to\infty\end{aligned}$$

Choose a large $k > 1$ such that $\text{meas}(\hat{E}_k) < \delta_0$, and then choose a $j \gg 1$ such that $\int_{\hat{E}_k} 4\pi r^2 \bar{f}(r, t_{n_j})\, dr > (1-\epsilon/4)[1-(T/T_b)^{3/5}]\, M_0$. Then we get a contradiction to (5.9): $V(f(\cdot, t_{n_j}), \delta_0) > (1-\epsilon/4)[1-(T/T_b)^{3/5}]\, M_0$. This proves (5.8).

In (5.8), for every $k \in \mathbf{N}$ (the set of all positive integers), choose $\delta = 2^{-k}$. Then for any $t \geqslant \tau(2^{-k})$, there exists a measurable set $\hat{E}_{k,t} \subset [0, \infty)$ with $\text{meas}(\hat{E}_{k,t}) < 2^{-k}$, such that

$$\int_{\hat{E}_{k,t}} 4\pi r^2 \bar{f}(r, t)\, dr > (1-\epsilon/2)\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right] M_0$$

Now let $t_0 = \max\{1, \tau(2^{-1})\}$,

$$n(t) = \max\{k \in \mathbf{N} \mid \max\{k, \tau(2^{-k})\} \leqslant t\} \qquad \text{for} \quad t \geqslant t_0$$

$$R = \sqrt{\frac{2M_2}{\epsilon[1-(T/T_b)^{3/5}]\, M_0}}$$

and define

$$S_t = \{v \in \mathbf{R}^3 \mid |v| \in \hat{E}_{n(t),t} \cap [0, R)\} \qquad \text{for} \quad t \geqslant t_0$$

Then $S_t \subset B_R$ and, since f conserves the energy, we have for all $t \geqslant t_0$

$$\begin{aligned}\int_{S_t} f(v, t)\, dv &= \int_{\hat{E}_{n(t),t} \cap [0,R)} 4\pi r^2 \bar{f}(r, t)\, dr > (1-\epsilon/2)\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right] M_0 - \frac{M_2}{R^2} \\ &= (1-\epsilon)\left[1-\left(\frac{T}{T_b}\right)^{3/5}\right] M_0\end{aligned}$$

Finally, since $\lim_{t\to\infty} n(t) = \infty$, it follows that for all $t \geqslant t_0$

$$\text{meas}(S_t) \leqslant 4\pi R^2\, \text{meas}(\hat{E}_{n(t),t}) \leqslant 4\pi R^2 2^{-n(t)} \to 0 \qquad (t \to \infty)$$

This proves the theorem. ∎

Remark. There leaves an important question concerning the behavior of the bounded concentration sets S_t: In Theorem 6, does the family $\{S_t\}_{t \geqslant t_0}$ can be so constructed that it is also monotonically non-increasing, i.e., $S_{t'} \supset S_t$ $\forall t_0 \leqslant t' < t < \infty$?

6. LONG-TIME BEHAVIOR(II): HIGH TEMPERATURE

In this section we will use the moment estimates (4.11) and (4.13) to give very high temperature conditions for solutions converging weakly to equilibrium as time tends to infinity. Because we hope to get quantitative results on the temperature conditions, the proof involves many detailed estimates using the positive constants A, A_s defined in (1.9) and (4.9) respectively.

Theorem 7. Suppose the kernel B satisfies (1.8)–(1.9). Let $f_0 \geqslant 0$ be an isotropic initial datum in $L^1_2(\mathbf{R}^3)$ satisfying $M_0 > 0$, and let f be the unique conservative isotropic solution of Eq. (BBE) with $f|_{t=0} = f_0$.

(i) ($\beta > 0$). Suppose M_0, M_2 satisfy a very high temperature condition:

$$\frac{T}{T_b} \geqslant \mathscr{A}(K^{3/(3-\beta)}\varepsilon M_0) \tag{6.1}$$

where

$$\mathscr{A}(y) = 8\left(\frac{2^{14}A}{A_{2+\beta}}\right)^{2(2-\beta)/\beta^2}\left(1+\frac{20}{A_0}\,y\right)^{2/\beta} y^{-2/3}, \qquad y > 0 \tag{6.2}$$

Then

$$f(v, t) \rightharpoonup \Omega_{a,b}(v) \text{ weakly in } L^1(\mathbf{R}^3) \text{ as } t \to \infty \tag{6.3}$$

where the coefficients a, b are the unique solution of the moment equation system (1.5) satisfying $0 < a \leqslant 1/\varepsilon$, $b > 0$.

(ii) ($\beta = 0$). Suppose $f_0 \in L^1_s(\mathbf{R}^3)$ for some $2 < s \leqslant 4$, and M_0, M_2 satisfy

$$K\varepsilon M_0 \leqslant \frac{A_0}{20}\left(\frac{A_s}{2^{4s+1}A}\right)^{2/(s-2)} \quad \text{and} \quad \frac{T}{T_b} \geqslant 8\left(\frac{A_s}{2^{4s-2}A}\right)^{2/s}(K\varepsilon M_0))^{-2/3} \tag{6.4}$$

Then the weak convergence (6.3) still holds.

Proof. First of all we note that since $\beta \leqslant 1$ and $A_s \leqslant A$ for $s \geqslant 0$, each of the conditions (6.2) and (6.4) implies $T/T_b > 1$. Thus by Proposition 1, the moment equation system (1.5) has a unique solution (a, b) satisfying

$0 < a \leqslant 1/\varepsilon$, $b > 0$. Next, by Theorem 4, in both cases (i) and (ii), we have for (some) $s > 2$, $\sup_{t \geqslant 1} \|f(\cdot, t)\|_{L^1_s(\mathbf{R}^3)} < \infty$. Since f conserves the mass and energy, this boundness and Theorem 5 imply that if there exists a time-sequence $\{t_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} t_n = \infty$ such that the sequence $\{f(\cdot, t_n)\}_{n=1}^{\infty}$ is weakly convergent in $L^1(\mathbf{R}^3)$, then the weak limit of $\{f(\cdot, t_n)\}_{n=1}^{\infty}$ must be the unique equilibrium solution $\Omega_{a,b}$ determined by the moment equation system (1.5). Thus if the set $\{f(\cdot, t)\}_{t\geqslant 0}$ is weakly compact in $L^1(\mathbf{R}^3)$, it must converge weakly in $L^1(\mathbf{R}^3)$ to $\Omega_{a,b}$ as $t \to \infty$. Therefore in the following we need only to prove the weak compactness of $\{f(\cdot, t)\}_{t\geqslant 0}$. Since f is an isotropic function in v and conserves the mass and energy, by Lemma 7 and the definition of $V(f(\cdot, t), \delta)$, it needs only to prove that $\sup_{t\geqslant 0} V(f(\cdot, t), \delta) \to 0$ as $\delta \to 0^+$.

We first give an estimate used for both cases (i) and (ii). For any $\delta > 0$ and any $E \in \mathscr{R}_\delta$, let $\phi(v) = 1_E(v)$ and $V_E(t) = \int_E f(v, t)\, dv$. By Lemma 8 and Proposition 2 we have with the constant $C_1 = C_1(f_0) > 0$

$$
\begin{aligned}
\frac{d}{dt} V_E(t) = {} & \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\{f'f'_*(1 + \varepsilon f + \varepsilon f_*) \\
& - ff_*(1 + \varepsilon f' + \varepsilon f'_*)\}\, \phi\, d\omega\, dv_*\, dv \\
\leqslant {} & \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, f'f'_* \phi\, d\omega\, dv_*\, dv \\
& + \varepsilon \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, f'f'_* f\phi\, d\omega\, dv_*\, dv \\
& + \varepsilon \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, f'f'_* f_*\phi\, d\omega\, dv_*\, dv \\
& - \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, f\phi f_*\, d\omega\, dv_*\, dv \\
\leqslant {} & C_1 \delta^{1/3} + 2\varepsilon K M_0^2 \int_{\mathbf{R}^3} f(v, t)\, \phi(v)\, dv + 8\varepsilon K M_0^2 V(f(\cdot, t), \delta) \\
& - \int_{\mathbf{R}^3} f(v, t)\, \phi(v) \left\{ \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, f(v_*, t)\, d\omega\, dv_* \right\} dv \\
\leqslant {} & C_1 \delta^{1/3} + 10\varepsilon K M_0^2\, V(f(\cdot, t), \delta) \\
& - \int_E f(v, t) \left\{ \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)\, f(v_*, t)\, d\omega\, dv_* \right\} dv
\end{aligned}
$$

For the negative term, using Lemma 10 we have

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f(v_*,t)\, d\omega\, dv_*$$

$$= \int_{\mathbf{R}^3} f(v_*,t) \left\{ \int_{\mathbf{S}^2} \min\{ K(\cos\theta)^2 \times \sin\theta\, |v-v_*|^3, b(\theta)\, |v-v_*|^\beta\}\, d\omega \right\} dv_*$$

$$\geqslant A_0 \int_{\mathbf{R}^3} f(v_*,t) \min\{K\, |v-v_*|^3, |v-v_*|^\beta\}\, dv_*$$

$$\geqslant \tfrac{1}{2} A_0 \int_{\mathbf{R}^3} f(v_*,t) \min\{K\, |v_*|^3, |v_*|^\beta\}\, dv_*$$

Thus, with

$$I_\beta(t) := \int_{\mathbf{R}^3} f(v_*,t) \min\{K\, |v_*|^3, |v_*|^\beta\}\, dv_*$$

we obtain

$$\frac{d}{dt} V_E(t) \leqslant C_1 \delta^{1/3} + 10\varepsilon K M_0^2\, V(f(\cdot,t),\delta) - \frac{1}{2} A_0 I_\beta(t)\, V_E(t), \qquad t \geqslant 0 \quad (6.5)$$

(i) $(\beta > 0)$. Since $\min\{K\, |v_*|^3, |v_*|^\beta\} = |v_*|^\beta - |v_*|^\beta (1 - K|v_*|^{3-\beta})^+ \geqslant |v_*|^\beta - K^{-\beta/(3-\beta)}$, we have

$$I_\beta(t) \geqslant M_\beta(f(\cdot,t)) - K^{-\beta/(3-\beta)} M_0$$

By (6.5), this gives for all $t \geqslant 0$

$$\frac{d}{dt} V_E(t) \leqslant C_1 \delta^{1/3} + \left(\frac{1}{2} A_0 K^{-\beta/(3-\beta)} M_0 + 10\varepsilon K M_0^2\right) V(f(\cdot,t),\delta) - \frac{1}{2} A_0 M_\beta(f(\cdot,t))\, V_E(t) \quad (6.6)$$

On the other hand, applying the conservation of energy and Hölder inequality we have

$$M_2 = M_2(f(\cdot,t)) \leqslant [M_{2+\beta}(f(\cdot,t))]^{1-\beta/2}\, [M_\beta(f(\cdot,t))]^{\beta/2}$$

In the moment estimates (4.11) (for $\beta > 0$), choose $s = 2 + \beta$ and choose $t_0 > 0$ such that $e^{at_0} = 2$. Then for $t \geqslant t_0$ we have $M_{2+\beta}(f(\cdot, t)) \leqslant M_2 2b_{2+\beta}$, where $b_{2+\beta}$ is given by (4.12) with $s = 2 + \beta$. This implies

$$M_\beta(f(\cdot, t)) \geqslant (2b_{2+\beta})^{-(2-\beta)/\beta} M_2, \qquad t \geqslant t_0$$

Therefore, by (6.6), we have

$$\frac{d}{dt} V_E(t) \leqslant C_1 \delta^{1/3} + C_2 V(f(\cdot, t), \delta) - C_3 V_E(t), \qquad t \geqslant t_0$$

where

$$C_2 = \tfrac{1}{2} A_0 K^{-\beta/(3-\beta)} M_0 + 10\varepsilon K M_0^2, \qquad C_3 = \tfrac{1}{2} A_0 \cdot (2b_{2+\beta})^{-(2-\beta)/\beta} M_2 \tag{6.7}$$

Now we prove that the condition

$$C_3 > C_2 \tag{6.8}$$

implies the L^1-weak compactness of $\{f(\cdot, t)\}_{t \geqslant 0}$. Suppose (6.8) holds. Then

$$\frac{d}{dt} [e^{C_3 t} V_E(t)] \leqslant e^{C_3 t} [C_1 \delta^{1/3} + C_2 V(f(\cdot, t), \delta)], \qquad t \geqslant t_0$$

This implies that for any $t \geqslant t_0$ and any $h > 0$,

$$\begin{aligned} &e^{C_3(t+h)} V(f(\cdot, t+h), \delta) \\ &\quad \leqslant e^{C_3 t} V(f(\cdot, t), \delta) + \int_t^{t+h} e^{C_3 \tau} [C_1 \delta^{1/3} + C_2 V(f(\cdot, \tau), \delta)]\, d\tau \end{aligned}$$

Since the function $t \mapsto V(f(\cdot, t), \delta)$ is Lipschitz continuous on $[0, \infty)$ (see (4.7)), it follows that

$$\frac{d}{dt} [e^{C_3 t} V(f(\cdot, t), \delta)] \leqslant e^{C_3 t} [C_1 \delta^{1/3} + C_2 V(f(\cdot, t), \delta)] \qquad \text{a.e.} \quad t > t_0$$

and therefore

$$\frac{d}{dt} V(f(\cdot, t), \delta) + (C_3 - C_2)\, V(f(\cdot, t), \delta) \leqslant C_1 \delta^{1/3} \qquad \text{a.e.} \quad t > t_0$$

Thus

$$V(f(\cdot,t),\delta)\leqslant V(f(\cdot,t_0),\delta)+\frac{C_1}{C_3-C_2}\,\delta^{1/3},\qquad t\geqslant t_0 \tag{6.9}$$

This together with Lemma 9 implies the weak compactness of $\{f(\cdot,t)\}_{t\geqslant 0}$.

Now we prove that the high temperature condition (6.1) implies (6.8). Note that according to (6.7), the condition (6.8) is equivalent to

$$\frac{M_2}{M_0}>(2b_{2+\beta})^{(2-\beta)/\beta}\left(K^{-\beta/(3-\beta)}+\frac{20\varepsilon K}{A_0}\,M_0\right) \tag{6.10}$$

Recalling (4.12) we have

$$\begin{aligned}(2b_{2+\beta})^{(2-\beta)/\beta}&=\left(\frac{M_2}{M_0}\right)^{(2-\beta)/2}\Bigg[\left(2K^{-\beta/(3-\beta)}+\frac{2^{4(2+\beta)+1}\varepsilon KM_0}{A_{2+\beta}}\right)\\&\quad\times(M_2/M_0)^{-\beta/2}+\frac{2^{4(2+\beta)+1}A}{A_{2+\beta}}\Bigg]^{(2-\beta)/\beta}\\&\leqslant\left(\frac{M_2}{M_0}\right)^{(2-\beta)/2}\Bigg[\left(2K^{-\beta/(3-\beta)}+\frac{2^{13}\varepsilon KM_0}{A_{2+\beta}}\right)\\&\quad\times(M_2/M_0)^{-\beta/2}+\frac{2^{13}A}{A_{2+\beta}}\Bigg]^{(2-\beta)/\beta}\end{aligned}$$

Thus, the following condition implies (6.10):

$$\begin{aligned}\frac{M_2}{M_0}&>\left[\left(2K^{-\beta/(3-\beta)}+\frac{2^{13}\varepsilon KM_0}{A_{2+\beta}}\right)(M_2/M_0)^{-\beta/2}+\frac{2^{13}A}{A_{2+\beta}}\right]^{2(2-\beta)/\beta^2}\\&\quad\times\left(K^{-\beta/(3-\beta)}+\frac{20\varepsilon K}{A_0}\,M_0\right)^{2/\beta}\end{aligned} \tag{6.11}$$

Let $T/T_b=\rho$. Then by equality (5.7) for T/T_b we have

$$\frac{M_2}{M_0}=c\rho(\varepsilon M_0)^{2/3}\qquad\text{with}\quad c:=\left(\frac{2\pi}{3}\,\frac{[\zeta(3/2)]^{5/3}}{\zeta(5/2)}\right)^{-1}>\frac{1}{8} \tag{6.12}$$

If we write $y = K^{3/(3-\beta)} \varepsilon M_0$ and

$$J(c\rho, y) = \left[\left(2y^{-\beta/3} + \frac{2^{13}}{A_{2+\beta}} \, y^{1-\beta/3} \right) (c\rho)^{-\beta/2} + \frac{2^{13} A}{A_{2+\beta}} \right]^{2(2-\beta)/\beta^2} \times \left(1 + \frac{20}{A_0} \, y \right)^{\beta/2} y^{-2/3}$$

then, by (6.12), the condition (6.11) is equivalent to

$$c\rho > J(c\rho, y) \tag{6.13}$$

Now suppose that M_0, M_2 satisfy the high temperature condition (6.1). Then since $c > 1/8$, we have, by (6.1)–(6.2),

$$\begin{aligned} c\rho &> \left(\frac{2^{14} A}{A_{2+\beta}} \right)^{2(2-\beta)/\beta^2} \left(1 + \frac{20}{A_0} \, y \right)^{\beta/2} y^{-2/3} \\ &= \left[\left(\frac{2^{14} A}{A_{2+\beta}} \right)^{(2-\beta)/\beta} \left(y^{-\beta/3} + \frac{20}{A_0} \, y^{1-\beta/3} \right) \right]^{2/\beta} \end{aligned}$$

This gives

$$J(c\rho, y) \leqslant \left[\frac{2y^{-\beta/3} + (2^{13}/A_{2+\beta}) \, y^{1-\beta/3}}{y^{-\beta/3} + (20/A_0) \, y^{1-\beta/3}} \left(\frac{A_{2+\beta}}{2^{14} A} \right)^{(2-\beta)/\beta} + \frac{2^{13} A}{A_{2+\beta}} \right]^{2(2-\beta)/\beta^2} \times \left(1 + \frac{20}{A_0} \, y \right)^{\beta/2} y^{-2/3}$$

Since $0 < \beta \leqslant 1$ and $A_s \leqslant A_0 \leqslant A$ for $s \geqslant 0$, we have

$$\frac{2y^{-\beta/3} + (2^{13}/A_{2+\beta}) \, y^{1-\beta/3}}{y^{-\beta/3} + (20/A_0) \, y^{1-\beta/3}} \left(\frac{A_{2+\beta}}{2^{14} A} \right)^{(2-\beta)/\beta} < \frac{2^{13} A}{A_{2+\beta}}$$

Thus we obtain (6.13):

$$J(c\rho, y) < \left(\frac{2^{14} A}{A_{2+\beta}} \right)^{2(2-\beta)/\beta^2} \left(1 + \frac{20}{A_0} \, y \right)^{\beta/2} y^{-2/3} < c\rho$$

and therefore the set $\{ f(\cdot, t) \}_{t \geqslant 0}$ is weakly compact in $L^1(\mathbf{R}^3)$.

(ii) ($\beta = 0$). In this case, the proof of the weak compactness is slightly different because for $\beta = 0$ the integral (see (6.5)) $I_0(t) \leqslant M_0$, so that in order to use $\frac{1}{2}A_0 \inf_{t\geqslant t_s} I_0(t)$ to suppress $10\varepsilon K M_0^2$, the first condition in (6.4) will be used. Choose $p=s/(s-2)$, $q=s/2$. Then by conservation of energy and Hölder inequality we have

$$M_2 \leqslant \left(\int_{\mathbf{R}^3} f(v_*, t) \min\{K\,|v_*|^3, 1\}\, dv_*\right)^{1/p}$$

$$\times \left(\int_{\mathbf{R}^3} f(v_*, t) \frac{|v_*|^{2q}}{[\min\{K\,|v_*|^3, 1\}]^{q/p}}\, dv_*\right)^{1/q}$$

i.e.,

$$M_2^{s/(s-2)} \leqslant I_0(t) \left(\int_{\mathbf{R}^3} f(v_*, t) \frac{|v_*|^s}{[\min\{K\,|v_*|^3, 1\}]^{(s-2)/2}}\, dv_*\right)^{2/(s-2)}$$

For the integral we have (since $2<s\leqslant 4$)

$$\begin{aligned}
&\int_{\mathbf{R}^3} f(v_*, t) \frac{|v_*|^s}{[\min\{K\,|v_*|^3, 1\}]^{(s-2)/2}}\, dv_* \\
&\quad = \int_{K\,|v_*|^3 \leqslant 1} f(v_*, t) \frac{|v_*|^s}{(K\,|v_*|^3)^{(s-2)/2}}\, dv_* \\
&\qquad + \int_{K\,|v_*|^3 > 1} f(v_*, t)\, |v_*|^s\, dv_* \\
&\quad \leqslant K^{-s/3} M_0 + M_s(f(\cdot, t))
\end{aligned}$$

Further, using the moment estimate (4.13) we have for some $t_s>0$

$$\begin{aligned}
&K^{-s/3} M_0 + M_s(f(\cdot, t)) \\
&\quad \leqslant \left(3K^{-s/3} + \frac{2^{4s}}{A_s}(A + K\varepsilon M_0)(M_2/M_0)^{s/2}\right) M_0, \qquad t \geqslant t_s
\end{aligned}$$

Thus, with

$$C_s := \left(3K^{-s/3} + \frac{2^{4s}}{A_s}(A + K\varepsilon M_0)(M_2/M_0)^{s/2}\right)^{-2/(s-2)} (M_0)^{-2/(s-2)}$$

we obtain

$$I_0(t) \geqslant (M_2)^{s/(s-2)}\, C_s, \qquad t \geqslant t_s$$

and so (6.5) for $\beta=0$ becomes

$$\frac{d}{dt} V_E(t) \leqslant C_1 \delta^{1/3} + 10\varepsilon K M_0^2 V(f(\cdot, t), \delta) - \frac{1}{2} A_0 (M_2)^{s/(s-2)} C_s V_E(t), \qquad t \geqslant t_s$$

As is shown in the case (i), to prove the weak compactness of $\{f(\cdot, t)\}_{t \geqslant 0}$, it needs only to prove that the condition (6.4) implies the condition

$$\tfrac{1}{2} A_0 (M_2)^{s/(s-2)} C_s > 10\varepsilon K M_0^2 \tag{6.14}$$

which gives an estimate of the form (6.9). The condition (6.14) is equivalent to the condition

$$\frac{M_2}{M_0} > \left(\frac{20K\varepsilon M_0}{A_0}\right)^{(s-2)/s} \left(3K^{-s/3} + \frac{2^{4s}}{A_s} (A + K\varepsilon M_0)(M_2/M_0)^{s/2}\right)^{2/s} \tag{6.15}$$

Write $T/T_b = \rho$ and $M_2/M_0 = (\varepsilon M_0)^{2/3}\, c\rho$ as above, and let $y = K\varepsilon M_0$. Then the condition (6.15) is equivalent to the condition

$$1 > \left(\frac{20y}{A_0}\right)^{(s-2)/s} \left(3y^{-s/3}(c\rho)^{-s/2} + \frac{2^{4s}}{A_s}(A+y)\right)^{2/s} \tag{6.16}$$

Now suppose M_0 and M_2 satisfy the condition (6.4), i.e.,

$$y \leqslant \frac{A_0}{20} \left(\frac{A_s}{2^{4s+1} A}\right)^{2s-2} \qquad \text{and} \qquad y^{-s/3}(c\rho)^{-s/2} \leqslant \frac{2^{4s-2} A}{A_s}$$

Then

$$3y^{-s/3}(c\rho)^{-s/2} + \frac{2^{4s}}{A_s}(A+y) < \frac{2^{4s+1} A}{A_s}$$

and so

$$\left(\frac{20y}{A_0}\right)^{(s-2)/s} \left(3y^{-s/3}(c\rho)^{-s/2} + \frac{2^{4s}}{A_s}(A+y)\right)^{2/s} < \left(\frac{20y}{A_0}\right)^{(s-2)/s} \left(\frac{2^{4s+1} A}{A_s}\right)^{2/s} \leqslant 1$$

Thus the condition (6.16) is satisfied and therefore the set $\{f(\cdot, t)\}_{t \geqslant 0}$ is weakly compact in $L^1(\mathbf{R}^3)$. This completes the proof. ∎

ACKNOWLEDGMENT

I am grateful to Professor Tianquan Chen for his encouragement and helpful discussions. This work was supported by the National Natural Science Foundation of China.

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